# CONSTRAINED NON-LINEAR SEISMIC INVERSION

TO EVALUATE A NET-TO-GROSS RATIO
OF A HYDROCARBON RESERVOIR

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A Thesis

Presented to

the Faculty of the Department of Earth and Atmospheric Sciences

University of Houston

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In Partial Fulfillment

of the Requirements for the Degree

Master of Science

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By

Aleksandar Jeremic

December 2008

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## Abstract

Theoretically, given gross reservoir thickness and seismic impedance of a binary sequence of pay and non-pay layers comprising the reservoir, it is possible to invert for net pay thickness if the properties of the layers are known. To determine the acoustic impedance, a post-stack constrained non-linear inversion that combines the random sampling technique is used.

It is found that first, an acoustic impedance of one of the layers has to be constrained to achieve unique solution. Second, Monte Carlo sampling technique allows convergence to the global minimum rather than local minimum in the optimization. Third, the greater the number of layers involved in the inversion, the closer estimated net-to-gross ratio is to the actual net-to-gross ratio. Finally, the inversion technique gives good estimation of the net-to-gross ratio when there is a good correlation between a net-to-gross ratio indicator and acoustic impedance, as it was found in South Timbalier field.

# Contents

# 1. Introduction

Implementing *a priori* information into an inversion algorithm can reduce the number of solutions and even cause the solution to be unique (Menke, 1984). *A priori* information can be seen as additional data regarding model parameters, which are acquired independently from the actual data. The actual data in this research include a window of a migrated seismic time section representing the normal incident response, whereas *a priori* information in the research includes acoustic impedance determined by well logging interpretation. In addition, parameters inferred by spectral inversion are included into the algorithm as constraints. They are not *a priori,* as they are derived from the data itself. However, they can be regarded as such by the inversion method, as a means of 1) biasing the inversion in directions found to be desirable for interpretation purposes, and 2) reducing the size of the initial-model space.

In the research, these two pieces of information are included into a non-linear inversion algorithm to produce acoustic impedance in time and, thus, combined with additional petrophysical information for a given layer, a net-to-gross ratio of a reservoir.

Synthetic wedge models are used to examine the effectiveness of such a non-linear inversion algorithm. In the examples, the synthetic data with various signal-to-noise ratios are inverted for model parameters, used to calculate the net-to-gross ratio of a layer. In a noise-free environment, the algorithm gives the deterministic solution – only one solution. However, in the presence of noise, for the same signal-to-noise ratio, different random noise is used and thus the algorithm gives the statistical solution in a form of histogram, standard deviation, and the mean value of predicted net-to-gross ratio.

Because the wedge model has assumed only three layers in the model and data to be inverted have been created from the actual parameterization of the inversion algorithm, another synthetic case is investigated. Here it is investigated the effects of the number of layers on the inversion result.

Finally, the same multi-layer algorithm is applied on a real dataset – South Timbalier field data.

There are not many examples in the literature that address the problem of determining the net-to-gross ratio of the layer with thickness below the seismic resolution. Most of the techniques used tried to estimate the gross thickness of the layer (Widess, 1973; Partyka et al., 1999; Marfurt and Kirlin, 2001; Castagna et al., 2003). The only attempt for solving the problem was using the spectral inversion (Partyka et al., 2006). This technique is actually also an inversion technique; however, it is in the frequency domain.

## 2. The quality of a hydrocarbon reservoir – the net-to-gross ratio

A net-to-gross ratio could be defined in different ways. In 1D case, petrophysicists assume that the net and gross thicknesses are dependant on the cut-off porosity. Thus, the total thickness of the reservoir with porosity greater then the cut-off is considered to be the net thickness, whereas the gross thickness is total thickness of the reservoir, including all values of porosity. However, because a sand-shale hydrocarbon reservoir is assumed here, usually net thickness is considered to be the thickness of sand, whereas gross thickness is the total thickness of the reservoir, including the shale and sand thickness (Figure 1).

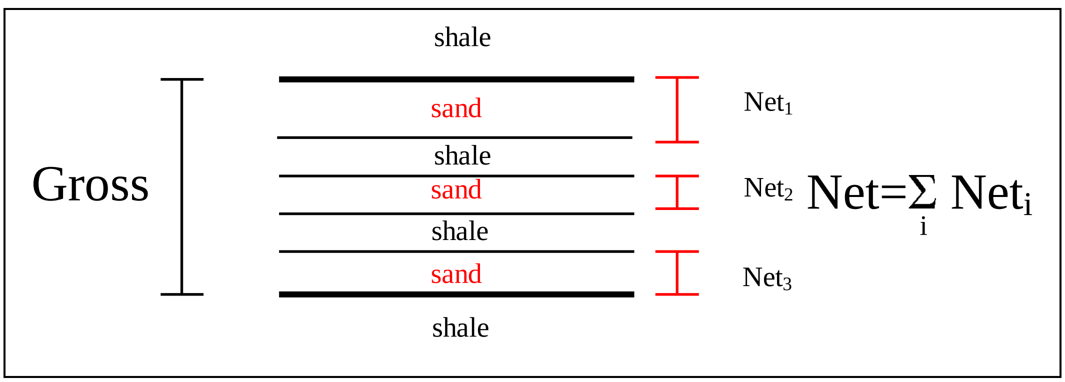



*Figure 1: Net vs. gross thickness in the sand-shale sequence (1D case).*

To examine the dependency of the quality of the reservoir on the acoustic impedance, there are two theories to be considered: ray theory and effective medium theory. These two theories correspond to the two so-called bounds: Voigt and Reuss, respectively.

## 2.1 Ray and effective medium theory

The ray theory is consistent with the observations if the wavelength ($\lambda$) of the wavelet dominant frequency is much less then the scale ($d$) of the layering ($\lambda << d$). In the ray theory, it is assumed that all constituents experience the same stress, representing the isostress or Voigt bound (Voigt, 1910). Furthermore, the effective elastic modulus ($M$) of the overall medium (in 1D case, stack of the layers), through which the wave propagates, is the arithmetic average of its constituent's moduli ($M_i$). Therefore, using the volume fractions ($f_i$):

$$M = \sum_i f_i M_i \tag{1}$$

On the other hand, the effective medium theory is consistent with the observations if the wavelength ($\lambda$) of the wavelet dominant frequency is much bigger then the scale ($d$) of layers ($\lambda >> d$). In the effective medium theory, it is assumed that all constituents experience the same strain, representing the isostrain or Reuss bound (Reuss, 1929). Moreover, the effective elastic modulus of the overall medium is the harmonic average of its constituent's moduli, often called Backus average (Backus, 1962). Thus:

$$M = \left( \sum_i \frac{f_i}{M_i} \right)^{-1} \tag{2}$$

However, the effective density of the medium ($\rho$) is the arithmetic average of its constituents' densities ($\rho_i$) in both theories:

$$\rho = \sum_i f_i \rho_i \tag{3}$$

Using the previous equations, the ray theory can be represented by the time-average or so-called Wyllie's equation (Wyllie et al., 1956):

$$\frac{1}{V} = \sum_i \frac{f_i}{V_i}, \text{ or } \Delta t = \sum_i \Delta t_i \tag{4}$$

whereas the effective medium theory can be represented by the following equation (Marion et al., 1994) :

$$\frac{1}{V^2} = \rho \sum_i \frac{f_i}{\rho_i V_i^2} \tag{5}$$

where $V$ is the effective seismic-wave velocity of the observed medium, $V_i$ are the seismic-wave velocities of the medium constituents, $\Delta t$ is the seismic travel time through the medium, and $\Delta t_i$ are the seismic travel time through the medium constituents.

These two simplest bounds or theories explain how the medium behaves in the extreme cases – very low and very high frequency compared to scale of the anisotropy. Defining these cases, all other media behavior is any combination of them. Therefore, applying these theories and determining the bounding net-to-gross ratios of the reservoir, through which the seismic wave propagates, give the range of all possible values of the net-to-gross of the reservoir.

## 2.2 Travel-time net-to-gross ratio – ray theory

Again, in 1D case (Figure 1), the net-to-gross ratio of the shale-sand reservoir is the ratio of the sand thickness ($h_{ss}$) and total thickness ($h$).

First, using the ray theory or starting from equations 3 and 4:

$$\rho = f_{sh}\rho_{sh} + f_{ss}\rho_{ss} \quad \text{and} \tag{6}$$

$$\frac{1}{V} = \frac{f_{sh}}{V_{sh}} + \frac{f_{ss}}{V_{ss}} \tag{7}$$

where *sh* and *ss* are subscripts representing the shale and sand constituent of the medium (reservoir). Because the normal incident wave is examined, that is 1D case, the volume fractions $f_i$ represents the fractions of the corresponding thickness: for shale, $f_{sh} = h_{sh} / h$ and for sand, $f_{ss} = h_{ss} / h$. Thus, the density equation can be rewritten in the following form:

$$\rho h = \rho_{sh} h_{sh} + \rho_{ss} h_{ss} \tag{8}$$

Because $h = V\Delta t$ where $V$ is seismic-wave velocity, the equation becomes:

$$\rho V \Delta t = \rho_{sh} V_{sh} \Delta t_{sh} + \rho_{ss} V_{ss} \Delta t_{ss} \tag{9}$$

Furthermore, as acoustic impedance $I$ is defined as $V\rho$ :

$$I\Delta t = I_{sh}\Delta t_{sh} + I_{ss}\Delta t_{ss} \tag{10}$$

Now substituting $\Delta t_{sh}$ using the time-average equation, $\Delta t_{sh} = \Delta t - \Delta t_{ss}$, in the previous equation:

$$I\Delta t = I_{sh}(\Delta t - \Delta t_{ss}) + I_{ss}\Delta t_{ss} \tag{11}$$

Finally:

$$\frac{\Delta t_{ss}}{\Delta t} = \frac{I - I_{sh}}{I_{ss} - I_{sh}} = Net / Gross_t \tag{12}$$

where $I$, $I_{ss}$, and $I_{sh}$ are P-wave acoustic impedances of the reservoir, sandstone, and shale, respectively.

Therefore, knowing acoustic impedance of a reservoir, together with acoustic impedances of clean sand and clean shale, the travel-time net-to-gross ratio can be estimated.

### 2.3 Travel-time net-to-gross ratio – effective medium

Now, the effective theory for shale-sand medium gives, equation 3 and 5:

$$\rho = f_{sh}\rho_{sh} + f_{ss}\rho_{ss} \text{ and} \tag{13}$$

$$\frac{1}{V^2} = \rho\left(\frac{f_{sh}}{\rho_{sh}V_{sh}^2} + \frac{f_{ss}}{\rho_{ss}V_{ss}^2}\right) \tag{14}$$

For 1D case, both equations can be rewritten in the following form:

$$\frac{h}{\rho V^2} = \frac{h_{sh}}{\rho_{sh}V_{sh}^2} + \frac{h_{ss}}{\rho_{ss}V_{ss}^2} \text{ and} \tag{15}$$

$$\rho h = \rho_{sh}h_{sh} + \rho_{ss}h_{ss} \tag{16}$$

and as $h = V\Delta t$, the equations becomes:

$$\frac{\Delta t}{\rho V} = \frac{\Delta t_{sh}}{\rho_{sh}V_{sh}} + \frac{\Delta t_{ss}}{\rho_{ss}V_{ss}} \text{ and} \tag{17}$$

$$\rho V\Delta t = \rho_{sh}V_{sh}\Delta t_{sh} + \rho_{ss}V_{ss}\Delta t_{ss} \tag{18}$$

Furthermore, introducing the acoustic impedance *I* as $v\rho$:

$$\frac{\Delta t}{I} = \frac{\Delta t_{sh}}{I_{sh}} + \frac{\Delta t_{ss}}{I_{ss}} \text{ and} \tag{19}$$

$$I\Delta t = I_{sh}\Delta t_{sh} + I_{ss}\Delta t_{ss} \tag{20}$$

Now substituting $\Delta t_{sh}$ of the first equation:

$$\Delta t_{sh} = I_{sh}\left(\frac{\Delta t}{I} - \frac{\Delta t_{ss}}{I_{ss}}\right) \quad (21)$$

into the second, the following equation is derived:

$$I\Delta t = I_{sh}^2 \frac{\Delta t}{I} - I_{sh}^2 \frac{\Delta t_{ss}}{I_{ss}} + I_{ss}\Delta t_{ss} \text{ or} \quad (22)$$

$$\left(I - \frac{I_{sh}^2}{I}\right)\Delta t = \left(I_{ss} - \frac{I_{sh}^2}{I_{ss}}\right)\Delta t_{ss} \quad (23)$$

Finally:

$$\frac{\Delta t_{ss}}{\Delta t} = \frac{I - \frac{I_{sh}^2}{I}}{I_{ss} - \frac{I_{sh}^2}{I_{ss}}} = \frac{I_{ss}\left(I^2 - I_{sh}^2\right)}{I\left(I_{ss}^2 - I_{sh}^2\right)} = Net/Gross_t \quad (24)$$

Using the effective medium theory, the same conclusion is derived: knowing acoustic impedance of a reservoir together with acoustic impedances of clean sand and clean shale, the travel-time net-to-gross ratio can be estimated.

### 2.4 Calculating reservoir travel-time net-to-gross ratio

One can easily prove that knowing the average acoustic impedance of the reservoir, $I_{mean}$, and clean sand and clean shale acoustic impedances, $I_{ss}$ and $I_{sh}$ is sufficient to determine the travel-time net-to-gross ratio of the reservoir; that is, the net-to-gross ratio determined from each infinitesimal layer *dt* (Figure 2) with acoustic impedance $I$ is directly related to the mean of the acoustic impedance $I_{mean}$. This argument stands for both Voigt and Reuss bounds. To prove that for the Voigt bound, using the equation 12:

$$Net/Gross_t = \frac{\int_{t_2}^{t_1} \frac{I - I_{sh}}{I_{ss} - I_{sh}} dt}{t_2 - t_1} \quad (25)$$

where $t_2$ is one-way travel time to the bottom of the reservoir and $t_1$ is one-way travel time to the top of the reservoir (Figure 2); therefore, $t_2 - t_1$ is the travel-time gross thickness. In a discrete form:

$$Net/Gross_t = \frac{\sum_i \frac{I_i - I_{sh}}{I_{ss} - I_{sh}} \Delta t}{n\Delta t} = \frac{\sum_i \frac{I_i - I_{sh}}{I_{ss} - I_{sh}}}{n} = \frac{\frac{\left(\sum_i I_i\right) - nI_{sh}}{I_{ss} - I_{sh}}}{n} =$$

$$= \frac{\frac{\left(\sum_i I_i\right) - nI_{sh}}{I_{ss} - I_{sh}}}{n} = \frac{\frac{\sum_i I_i}{n} - I_{sh}}{I_{ss} - I_{sh}} = \frac{I_{mean} - I_{sh}}{I_{ss} - I_{sh}} \quad (26)$$

Therefore, to calculate travel-time net-to-gross only the mean value of the acoustic impedance of the reservoir is sufficient. Now the same can be applied for the Reuss bound; thus, starting from equation 24:

$$Net/Gross_t = \frac{\int_{t_2}^{t_1} \frac{I_{ss}\left(I^2 - I_{sh}^2\right)}{I\left(I_{ss}^2 - I_{sh}^2\right)} dt}{t_2 - t_1} \quad (27)$$

in the discrete form:

$$Net/Gross_t = \frac{\sum_i \frac{I_{ss}\left(I_i^2 - I_{sh}^2\right)}{I\left(I_{ss}^2 - I_{sh}^2\right)} \Delta t}{n\Delta t} = \frac{\sum_i \frac{I_{ss}\left(I_i^2 - I_{sh}^2\right)}{I_i\left(I_{ss}^2 - I_{sh}^2\right)}}{n} = \frac{\frac{I_{ss}}{\left(I_{ss}^2 - I_{sh}^2\right)}\left(\sum_i \frac{I_i^2}{I_i} - \sum_i \frac{I_{sh}^2}{I_i}\right)}{n} =$$

$$= \frac{\frac{I_{ss}}{\left(I_{ss}^2 - I_{sh}^2\right)}\left(\sum_i I_i - \frac{I_{sh}^2}{\sum_I I_i}\right)}{n} = \frac{I_{ss}}{\left(I_{ss}^2 - I_{sh}^2\right)}\left(\frac{\sum_i I_i}{n} - \frac{I_{sh}^2}{\frac{\sum_I I_i}{n}}\right) = \frac{I_{ss}\left(I_{mean}^2 - I_{sh}^2\right)}{I_{mean}\left(I_{ss}^2 - I_{sh}^2\right)} \quad (28)$$

Again, the same conclusion can be made: the knowing of mean value of the acoustic impedance of the reservoir suffices to determine the travel-time net-to-gross ratio of the reservoir.

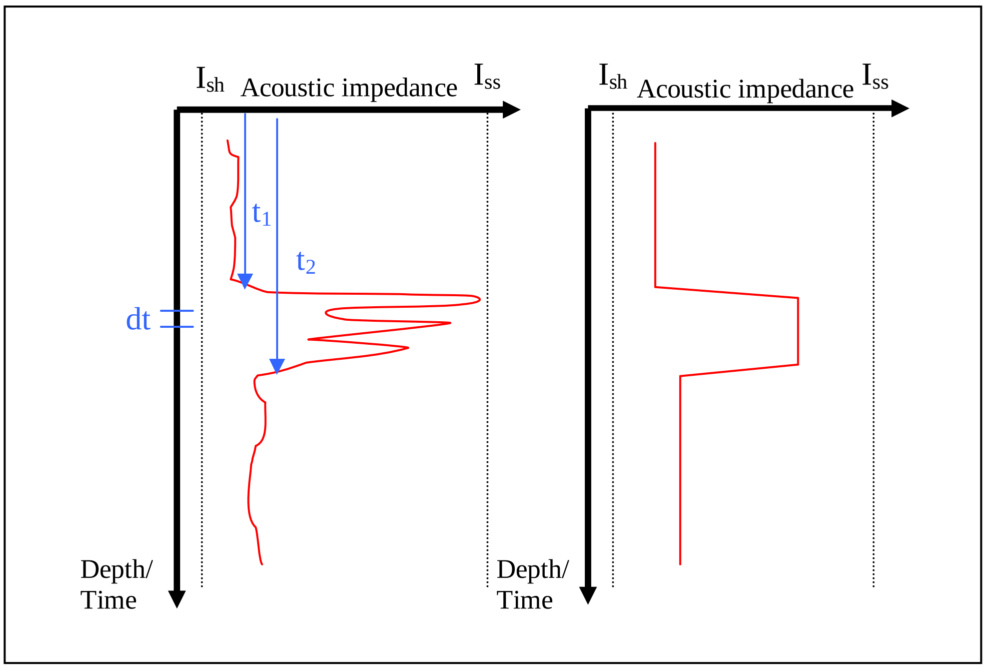



*Figure 2: Real and average acoustic impedances – the same net-to-gross ratio.*

In this research, acoustic impedance of the reservoir is going to be estimated using non-linear inversion, whereas the acoustic impedances of the clean sandstone and shale are going to be estimated from the well logging measurement. The output of the inversion is not the average thickness of the reservoir but the starting acoustic impedance of the reservoir $I_0$ and gradient of the acoustic impedance within the reservoir $g$, so that the average can be easily calculated:

$$I_{mean} = I_0 + \frac{g(1+2+...+(n-1))}{n} \tag{29}$$

where $n$ is the travel-time thickness of the reservoir.

## 2.5 True net-to-gross ratio

Now, to calculate net-to-gross ratio in depth domain, that is, true net-to-gross ratio, the ratio between the velocity of clean sand $V_{ss}$ and the velocity of reservoir $V$ is needed:

For ray theory:

$$Net/Gross = \frac{h_{ss}}{h} = \frac{\Delta t_{ss} V_{ss}}{\Delta t V} = \frac{I - I_{sh}}{I_{ss} - I_{sh}} \frac{V_{ss}}{V} = Net/Gross_t \frac{V_{ss}}{V} \tag{30}$$

For effective medium theory:

$$Net/Gross = \frac{h_{ss}}{h} = \frac{\Delta t_{ss} V_{ss}}{\Delta t V} = \frac{I_{ss}\left(I^2 - I_{sh}^2\right)}{I\left(I_{ss}^2 - I_{sh}^2\right)} \frac{V_{ss}}{V} = Net/Gross_t \frac{V_{ss}}{V} \tag{31}$$

Hence, if the ratio of clean sand velocity and velocity of reservoir is close to one, the travel-time net-to-gross ratio could be very accurate in the estimation of the economic value of the reservoir.

## 3. Theory of seismic inversion

To study any physical system, three elements are to be included (Tarantola, 2005):

1) Forward modeling – which uses the discovered physical laws on given values of model parameters to predict the observable data (Figure 3).

2) Parameterization of the system – which includes discovery of the minimal sets of the model parameters, whose values completely characterize the system.

3) Inverse modeling – which uses the measurements of the observable data to estimate the actual values of the model parameters (Figure 3).

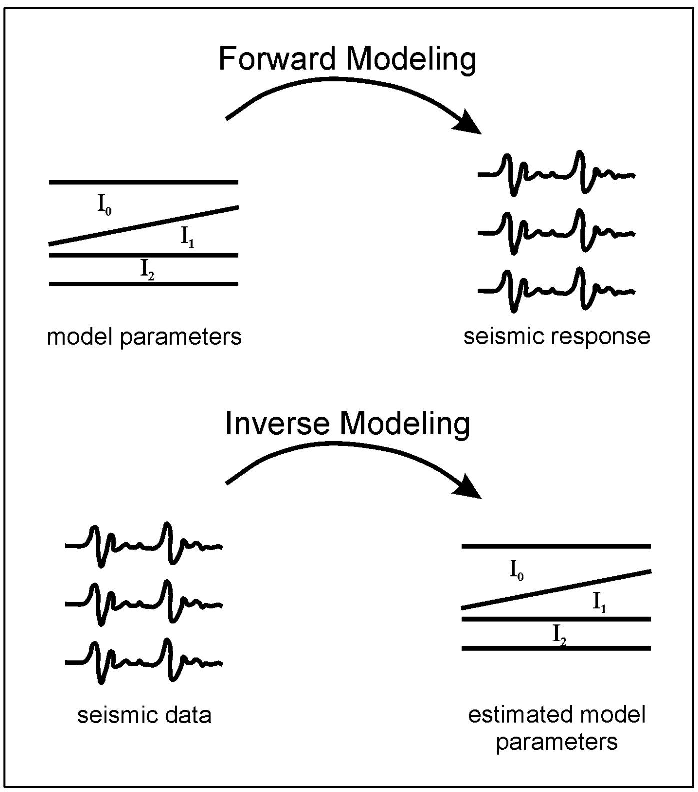



*Figure 3: Forward vs. inverse modeling (Treitel et al., 1993).*

Forward modeling and inverse modeling (or inversion) are two opposite processes (Russell, 1988): forward modeling is a procedure for creating model data from the known model parameters; conversely, inverse modeling or inversion is a procedure for extracting model parameters from the acquired data. Consequently, two spaces can be defined: model space and data space. Thus, model parameters represent a vector in the model space, usually denoted by *m*, whereas seismic data represent a vector in the data space, usually denoted by *d*:

$$m = [m_1 \quad m_2 \quad \dots \quad m_{M-1} \quad m_M]^T$$

$$d = [d_1 \quad d_2 \quad \dots \quad d_{N-1} \quad d_N]^T$$

where M and N are dimensions of the model and data spaces, respectively.

### 3.1 Forward modeling

To develop a forward modeling algorithm in a noise-free environment, a seismic data in time domain $d_t$ can be regarded as convolution of a seismic wavelet $w_t$ and reflectivity function $r_t$:

$$d_t = \sum_{\tau} w_{\tau} r_{t-\tau} \tag{32}$$

For normal incident waves, pressure reflection coefficients depend only on acoustic impedances of media $I = \rho V$ :

$$r_j = \frac{I_{j+1} - I_j}{I_{j+1} + I_j} = \frac{\rho_{j+1} V_{j+1} - \rho_j V_j}{\rho_{j+1} V_{j+1} + \rho_j V_j} \tag{33}$$

where $\rho$ is density, and $V$ is P-wave velocity of a layer.

Using the previous equations, one can easily develop an algorithm for seismic forward modeling: knowing the model parameters, P-wave acoustic impedances with time, and the source signature and combining the equations 13 and 14, a synthetic seismic trace can be produced.

To use this forward model, there are many assumptions. First, the effects of 1) geometrical spreading, 2) transmission, and 3) all multiples have been removed and properly compensated in the given seismic data. Second, the seismic trace is calibrated in such a way that the amplitude of the seismic data represents the absolute values corresponding to the reflection coefficients. Third, the data are noise-free.

### 3.2 Parameterization

A very important part to be included into the inversion and forward modeling concepts is how the model parameters are defined to completely characterize the system – in this case, the model is defined by the P-wave acoustic impedance profile and the source signature. The acoustic impedance profile is represented using so called discrete interval parameterization (Cooke and Schnieder, 1983). This type of parameterization involves three parameters for each of $L$ layers in the profile: (1) starting acoustic impedance $I_i$, (2) two-way travel-time thickness of layers $\Delta t_i$, and (3) acoustic impedance gradient $g_i$ (Figure 4), where $i = 1, 2, 3 \ldots L$ (Figure 4).

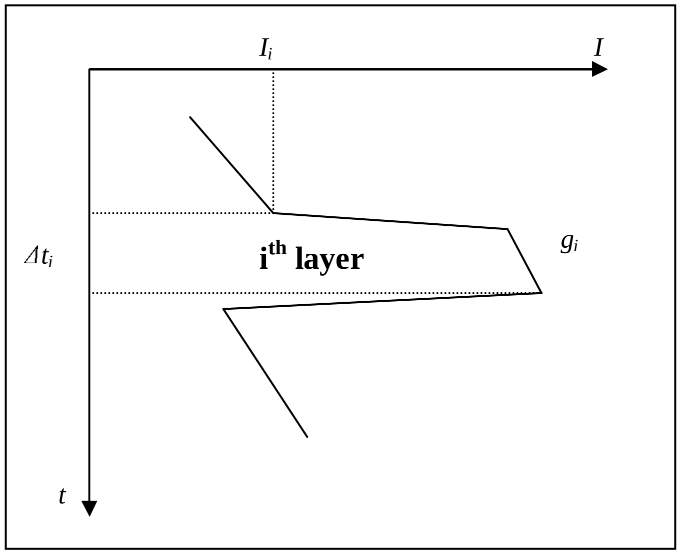



*Figure 4: Discrete interval parameterization of acoustic impedance with time.*

The source signature is considered to be Ricker wavelet (Figure 5). Ricker wavelet is a zero-phase wavelet representing a second derivative of the Gaussian function or the third derivative of the normal probability density function (Sheriff, 2002) To define Ricker wavelet, only one parameter is needed – the dominant frequency $f_M$. Thus, in time domain, the equation for the Ricker wavelet is the following:

$$f(t) = \left(1 - 2\pi^2 f_M^2 t^2\right) e^{-\pi^2 f_M^2 t^2} \tag{34}$$

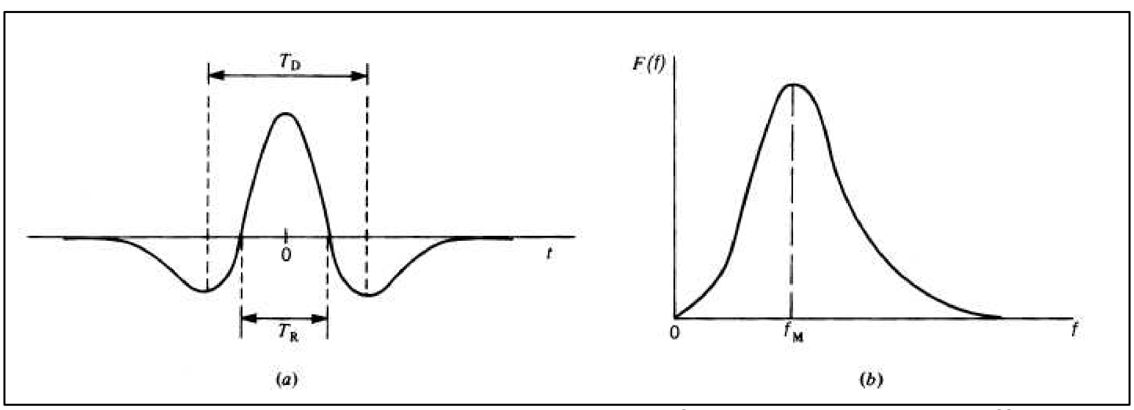



*Figure 5: Ricker wavelet - a) time domain and b) frequency domain (Sheriff, 2002).*

The dominant frequency can be related to mean frequency $f_{mean}$:

$$f_{mean} = \left(\frac{2}{\pi^{1/2}}\right) f_M \tag{35}$$

Now when the parameterization has been defined, a dimension of the model space and thus model vector can be determined. If the number of layers is $L$, by constraining the travel-time thickness of one layer, two of $L$ travel-time thicknesses are dependent on the other $L-2$ travel-time thicknesses; thus, because a source signature is defined by one parameter (dominant frequency), the number of $3*L-1$ parameters is needed to completely characterized the physical system:

$$m = [I_1 I_2 \cdots I_L g_1 g_2 \cdots g_L \Delta t_1 \Delta t_2 \cdots \Delta t_{L-2}\, f_M]^T$$

The only problem left regarding parameterization is the number of layers ($L$) used in the inversion. To estimate number of layers, that is, actually to estimate the number of degrees of freedom, the applied algorithm uses a scanning technique. The different number of layers is investigated; thus, with increasing the number of layers, when the average of the acoustic impedance of the layer of interest becomes stable, it suggests that the degrees of freedom are good enough for the purpose of the estimating net-to-gross ratio. The detailed analysis is given in the Section 6: Synthetic example – Multi-layer case.

### 3.3 Non-linear inversion

Now, when the forward model and parameterization are defined, the inversion procedure should be determined. Generally, there are two types of the inversion: linear and non-linear. As their names imply, linear and non-linear inversions solve for the model parameters that linearly and non-linearly affect the data, respectively.

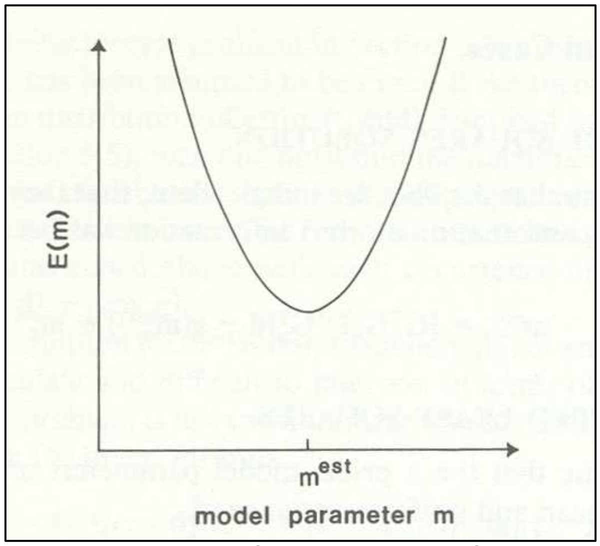



*Figure 6: One minimum in the objective function - E(m) of a linear model parameter – m (Menke, 1984).*

Both linear and non-linear inversion algorithms are usually solved by the optimization of an objective function. This function is most commonly the sum of the squares errors of observed and predicted data although it can be any kind of a norm function. The least squares technique is most popular because its solution represents the maximum likelihood solution, if the data errors follow Normal (Gaussian) distribution (Menke, 1984).

Using any optimization technique, the unique solution of the inversion should be the global minimum of the function. Furthermore, the objective function of a linear problem shows only one minimum (Figure 6), whereas the objective function of a non-linear problem can have one global as well as local minimum (Figure 7). This difference in the complexity of the objective functions forces geophysicists to treat these problems separately.

As it can be seen from the forward convolution modeling (Equations 32 and 33), the acoustic impedances and wavelet parameters are related non-linearly, that is, they non-linearly affect data. Therefore, the non-linear inversion must be used to solve for the parameters.

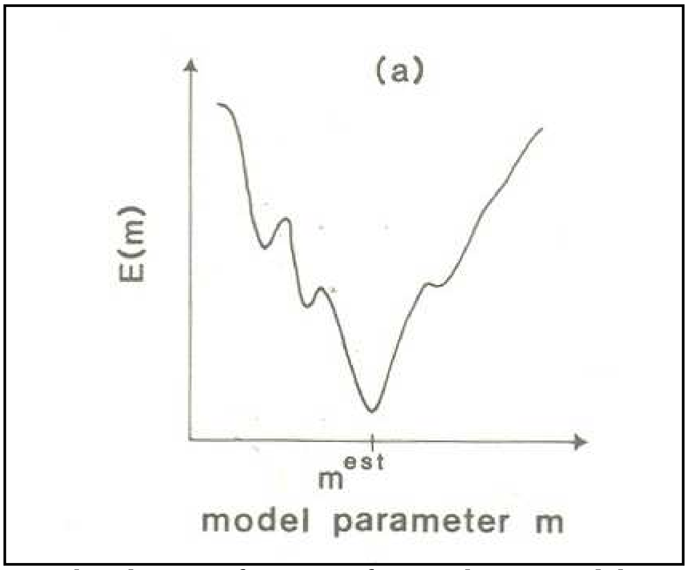



*Figure 7: The objective function of a non-linear model parameter could have one global minimum and local minima (Menke, 1984).*

There are many different approaches to solve non-linear inverse problems. One of the non-linear inversion techniques is an iterative procedure using Taylor series expansion and forward model algorithm to extract the model parameters. This technique is often called Generalize Linear Inversion. It consists of representing any seismic data *d(m)* for which parameters *m* should be solved for in terms of a synthetic seismic trace *d(m')* for which model parameters *m'*, called initial model parameters, are known:

$$d(m) = d(m') + \frac{\partial d(m')}{\partial m'}(m - m') + \frac{\partial^2 d(m')}{\partial m'^2}\frac{(m - m')^2}{2!} + \ldots \qquad (36)$$

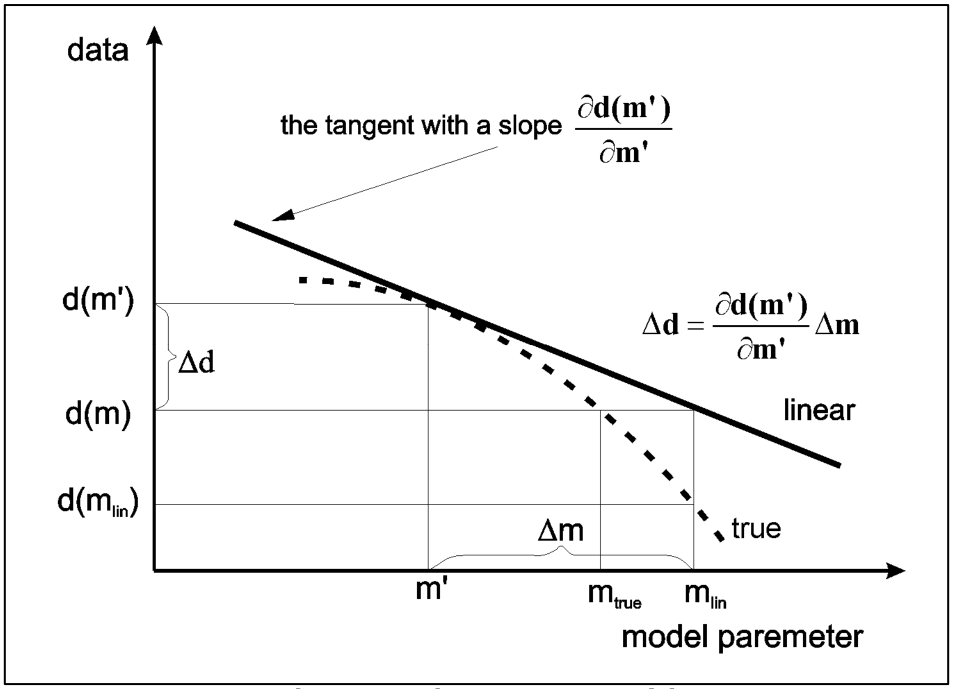



*Figure 8: Taylor series – linear vs. true model parameters.*

Assuming that any function, non-linear in this case, shows linear character in the neighborhood of *d(m)* and putting *d(m')* on a left-hand side, the previous equation can be simplified and easily depicted graphically (Figure 8):

$$d(m) - d(m') = \frac{\partial d(m')}{\partial m'}(m_{lin} - m') \tag{37}$$

This equation can be solved for $(m_{lin}-m')$ and shown in a matrix form:

$$\begin{bmatrix} m_1 - m_1' \\ m_2 - m_2' \\ \vdots \\ m_M - m_M' \end{bmatrix} = \begin{bmatrix} \frac{\partial d_1(m_1')}{\partial m_1'} & \frac{\partial d_1(m_2')}{\partial m_1'} & \cdots & \cdots \\ \frac{\partial d_2(m_1')}{\partial m_1'} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \frac{\partial d_{N-1}(m_M')}{\partial m_M'} \\ \cdots & \cdots & \frac{\partial d_N(m_{M-1}')}{\partial m_{M-1}'} & \frac{\partial d_N(m_M')}{\partial m_M'} \end{bmatrix}^{-1} \begin{bmatrix} d_1(m) - d_1(m') \\ d_2(m) - d_2(m') \\ \vdots \\ d_N(m) - d_N(m') \end{bmatrix} \tag{38}$$

where *M* is a number of model parameters and *N* is a number of data. Simplified:

$$\Delta m = J_m^{-1} \Delta d \tag{39}$$

where $J_m$ is an *MxN* matrix, known as a sensitivity matrix or Jacobian.

To avoid ill-posed solution of the estimated model parameters, a geophysicist must consider the relationship between a number of data (*N*), a number of model parameters (*M*), and a number of linearly independent pieces of information in a kernel matrix or Jacobian (*P*) (Richardson and Zandt, 2005). Therefore, there are four possibilities - classes:

1) Class I: $P=M=N$ – often called evendetermined problem. This type of problem has one unique solution and predicted model exactly fit the data.
2) Class II: $P=M<N$ – often called overdetermined problem, which is mostly present in geophysics practice. This type of problem has only one solution and the predicted model does not exactly fit the data.
3) Class III: $P=N<M$ – often called underdetermined problem, which is also used in geophysics practice. This type of problem gives an infinite number of model parameters solutions, which exactly fit the data. Usually, to form a unique solution, another criterion is needed. For example, the solution must have a minimum length.
4) Class IV: $P<M<N$, $P<N<M$ or $P<M=N$. Both model and data space have higher dimension then the number of linearly independent pieces of information, making the problem ill – posed. Non-uniqueness exists in both directions – data space and model space.

In this research procedure, the number of data (N) and model parameters (M), and a rank of Jacobian (P) makes class II – overdetermined problem, which is accomplished by the following:

1) choosing the number of data to be greater then model parameters, and

2) constraining one model parameter – starting acoustic impedance of one of the layers

Actually, two model parameters are constrained, here. However, only one is important in terms of the solving the non-uniqueness problem, which is going to be explained in Section 3.4 – Non-linear inversion including constraints.

Furthermore, in the class II problem, minimizing the objective function – $|\Delta d(m)-J\Delta m|^2$, the least-squares solution, often called a Gauss-Newton solution, is derived:

$$\Delta m = (J_m^T J_m)^{-1} J_m^T \Delta d \tag{40}$$

Finally, model parameters are estimated using the initial model parameters $m'$ and the solution of the previous least-squares technique $\Delta m$:

$$m_{lin} = m' + \Delta m \tag{41}$$

The model parameters $m_{lin}$ (Figure 8) would represent the estimated model parameters only if 1) the model parameters linearly affected in the neighborhood $d(m)$ and $d(m')$ and/or 2) a least-squares error $\sum (d(m) - d(m_{lin}))^2$ had a satisfactory small value. If these two previous conditions are not satisfied, iteratively the error can be reduced, where the estimated parameters $m_{lin}$ become new initial model parameters $m'$.

Thus, a non-linear inversion usually starts with defining the initial model, and the inversion algorithm solution iteratively converges to the neighboring minimum or even maximum, so that the solution of this type of non-linear inversion algorithm could be not only the desired global minimum but also a local minimum or even a maximum (Figure 7). Consequently, the mathematical solution of the optimization could be any of these minima and maxima. One of the ways to find a global minimum instead of a local minimum or maximum is to search model space by random sampling – often called Monte Carlo sampling. In addition, using the constraints, the number of dimensions in the searching model space is reduced.

### 3.4 Non-linear inversion including constraints

The previous analysis represents an unconstrained least squares or Gauss-Newton solution of the inverse problem, and it could be used if no other information regarding model parameters is available. If additional pieces of information regarding some parameters are known, for example, their relations or their values, they can be implemented and thus can improve the least-squares minimization.

The following matrix relation can represent the general form of equality relations:

$$Fm = k \tag{42}$$

where $F$ is a $KxM$ matrix to be formed according to the additional information , $m$ is a $M$-dimensional vector of model parameters, and $k$ is a $K$-dimensional vector to be formed according to the additional information as well. ($M$ is the number of model parameters and $K$ is the number of additional relations, that is, the number of constraints.)

Thus, for example, if travel-time thickness of two beds determined by other method are denoted by $t_0$ and $t_2$ and because the initial travel-time thickness for these two beds are the same, denoted by $t_0^{'}$ and $t_2^{'}$, the corresponding perturbations should be zero:

$$\Delta t_0 = 0 \text{ and } \Delta t_2 = 0 \tag{43}$$

or in the matrix form:

$$F\Delta t = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \Delta t_1 \\ \Delta t_2 \\ \Delta t_3 \\ \Delta t_4 \\ \vdots \\ \Delta t_{L-2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_3 \end{bmatrix} \quad (44)$$

One of the ways to constrain the equality relations regarding model parameters is to use Lagrange multipliers. Lagrange multiplier – λ is a scaling factor between two vectors – the gradient of objective function and the gradient of an equation that defines additional information to be incorporated into the optimization process (Figure 9).

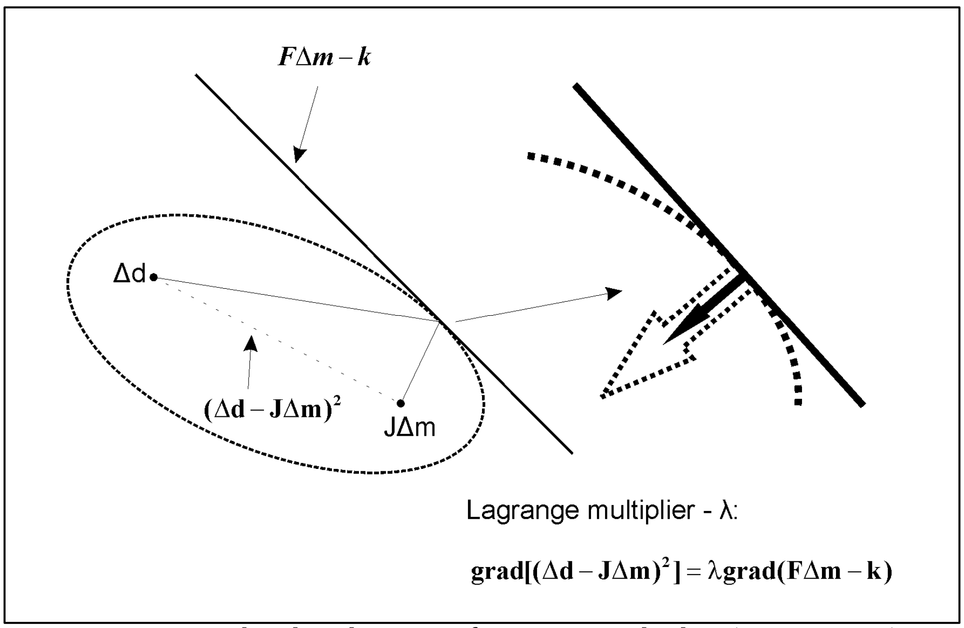



*Figure 9: Graphical explanation of Lagrange multiplier (Jensen, 2004).*

To derive the physical meaning of the Lagrange multiplier, first, it is clear that the obtained minimum (or maximum) of the objective function is a function of the values used to constrained the given parameter, that is, the value of *k*; thus, such a function can be defined as follows (Karabulut H., 2006):

$$R(k) = E_{\min}(m,k) \quad (45)$$

where the function *R(k)* is the value of the obtain minimum (or maximum) of the objective function, satisfying the given constraints.

Now, the derivative or gradient of the previous function with the respect of the vector *k* is going to be the following:

$$grad_k R(k) = \frac{\partial R(k)}{\partial k} = \frac{\partial E_{\min}(m,k)}{\partial m}\frac{dm}{dk} = grad_m(E_{\min}(m,k))\frac{dm}{dk} \qquad (46)$$

From Figure 9:

$$grad_m E(m,k) = \lambda\, grad_m (Fm - k) \qquad (47)$$

Therefore:

$$grad_k R(k) = \lambda \cdot grad_m(Fm-k)\frac{dm}{dk} = \lambda \cdot \frac{\partial(Fm-k)}{\partial m}\frac{dm}{dk} = \lambda \cdot \left(F - \frac{\partial k}{\partial m}\right)\frac{dm}{dk} \qquad (48)$$

Finally, as *F* is not function of *k*, then:

$$grad_k R(k) = -\lambda \frac{\partial k}{\partial m}\frac{dm}{dk} = -\lambda \qquad (49)$$

Therefore, the multiplier is the derivative of the obtained minimum (or maximum) of the objective function with the respect to the constraint value. Again:

$$\lambda = -\frac{\partial R(k)}{\partial k} = -\frac{\partial E_{\min}(m,k)}{\partial k} \qquad (50)$$

There are few interpretations to be made about the previous simple relation:

1) If $\lambda$ is close to zero, the change in constraint value (*k*) has not affected the obtained minimum of the objective function (*E*).

2) If $\lambda$ is constant with changing *k*, it suggests that the parameter involved in the constraint equation behaves as an *annihilator*, making the solution non-unique; that is,

the objective function is insensitive to its change. Therefore, such a parameter should be constrained by *a priori* information in order to solution be unique.

2) On the other hand, if $\lambda$ is not constant, the change in $k$ has affected the value of the objective-function minimum. The sensitiveness of the objective function on a change in the $k$ value of the constraint equality equation suggests that if there is a unique $k$ value that is close to zero, then the constrained parameter does not have to be defined by *a priori* information and the solution is unique. In other words, the solution of the constrained inversion will correspond to the unconstrained minimum of the objective function. In addition, in the case where the bigger the $\lambda$ becomes, the higher the force of the constraint.

In conclusion, Lagrange multiplier is a kind of indicator if the parameter should be constrained or not: if it has constant value with changing the constrained value of the parameter, the parameter must be constrained to have unique solution of the inversion, or if it has a minimum (preferably close to zero), which corresponds to unique constrained value of the parameter, then the parameter does not have to be constrained to have a unique solution. This usage of Lagrange multiplier was investigated in the chapter 5 – Wedge models. The application of such a Lagrange multiplier interpretation is investigated only on the wedge synthetic models.

Now, how to implement the constraints with Lagrange multipliers technique? Using the relation involving Lagrange multipliers (Figure 9) together with the least-squares optimization, the following solution is derived (Menke, 1984):

$$\begin{bmatrix} \Delta m \\ \lambda \end{bmatrix} = \begin{bmatrix} J_m^T J_m & F^T \\ F & 0 \end{bmatrix}^{-1} \begin{bmatrix} J_m^T \Delta d \\ k \end{bmatrix} \qquad (51)$$

In addition, to ensure the convergence of the inversion solution into the neighboring minimum, the Gauss-Newton solution can be modified into Marquardt-Levenberg solution, which implements the following equality equation into the solution:

$$\beta \Delta m^T \Delta m = 0 \qquad (52)$$

Lagrange multiplier $\beta$ is often called damping factor. It prevents unbounded oscillations in the solution, that is, smooth the model parameter change vector $\Delta m$ (Treitel et al., 1993). Using the equation 49, Marquardt-Levenberg solution of the least-squares optimization becomes:

$$\Delta m = (J_m^T J_m + \beta I)^{-1} J_m^T \Delta d \qquad (53)$$

where $I$ is an $MxM$ identity matrix

Finally, Marquardt-Levenberg least-squares minimization implementing additional information in the form $F\Delta m = k$ is the following:

$$\begin{bmatrix} \Delta m \\ \lambda \end{bmatrix} = \begin{bmatrix} J_m^T J_m + \beta I & F^T \\ F & 0 \end{bmatrix}^{-1} \begin{bmatrix} J_m^T \Delta d \\ k \end{bmatrix} \qquad (54)$$

In addition to this algorithm, if again the $L$ is the number of layers, the additional information on two of $3L$ model parameters are constrained by the previously explained technique: the starting acoustic impedance of one layer – $I_y$ and the two-way travel-time thickness of the layer of interest – $\Delta t_x$. Both constraints are implemented using the equality equations in the matrix form explained previously $- F\Delta m = k$. As their values are considered to be known and determined by other method, the perturbation model values should stay zero. Thus, as:

$$m = [I_1 I_2 \cdots I_y \cdots I_L g_1 g_2 \cdots g_L \Delta t_1 \Delta t_2 \cdots \Delta t_x \cdots \Delta t_{L-2}\, f_M]^T$$

$$\Delta m_y = 0 \text{ and } \Delta m_{2L+x} = 0 \tag{55}$$

therefore:

$$F\Delta m = \begin{bmatrix} 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \Delta m_1 \\ \Delta m_2 \\ \Delta m_3 \\ \Delta m_4 \\ \vdots \\ \Delta m_{3L-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} k_y \\ k_{2L+x} \end{bmatrix} \tag{56}$$

To improve the Newton-Gauss solution, the damping factor $\beta$ should be implemented into the optimization. If the damping factor is zero, it is clear from equation 50 that Marquardt-Levenberg solution becomes Newton-Gauss solution. However if $\beta$ is infinity, the Marquardt-Levenberg solution becomes so called the steepest descent method (Madsen et al., 2004). Therefore, the Marquardt-Levenberg method is sometimes called hybrid method.

Because the steepest descent method gives the better estimation of optimization solution, if the initial model is far from the minimum, and as the Gauss-Newton solution gives much faster convergence if the initial model is close to the minimum, the $\beta$ factor should be adopted accordingly (Marquardt, 1963).

### 3.5 Non-linear inversion using random sampling

Monte Carlo methods are used to find mathematical solutions of the problems that cannot be easily found by the analytical or other numerical methods. The implementation of Monte Carlo method is especially desired when the space dimensions of the problem increases. The Monte Carlo method in solving the seismic inversion problems was previously introduced by Keilis-Borok and Yanovskaya (1967) and Press (1968, 1971).

Here, the Monte Carlo method helps in finding model parameters that corresponds to the global rather then local minimum of the objective function – in this case, a sum of the least-squares between two traces. In other words, if the initial model of the inversion algorithm is uncertain and unpredictable but the probability density function of the model parameters could be estimated or reliably assumed, the global minimum of the inversion algorithm could be found by Monte Carlo random sampling.

Before explaining the Monte Carlo random sampling technique, probability density and cumulative distribution functions must be defined. Any random variable *x* has a probability density function *f(x)* so that *f(x)dx* represents the probability that the random variable ***x*** has a value between *x* and *x+dx*. Consequently:

$$\int_{-\infty}^{+\infty} f(x)dx = 1, \text{ and } f(x)dx \geq 0 \tag{57}$$

If the probability density function is *f(x)* and the corresponding cumulative density function is *F(x)*, then:

$$F(x) = \int_{-\infty}^{x} f(x)dx \tag{58}$$

Now, to sample model space from a given probability density function, there are a couple of different methods to be used. Here, the inverse transform method is used (Von Neumann, 1947). It is sometimes called “Golden Role for Sampling”. To implement this technique of the random sampling, three steps are needed:

1) Sample a number $\xi$ from a random number generator creating the uniformly independent values on [0,1] interval.

2) Equate ξ with the cumulative distribution function: $F(x) = \xi$.

3) Invert the cumulative distribution function and solve for $x$: $x = F^{-1}(\xi)$.

The $x$ value determined in such a way represents a sample of the random variable being consistent with its probability density function $f(x)$. This inversion procedure is not always feasible. However in the case where the uniform probability density function is used, it is possible.

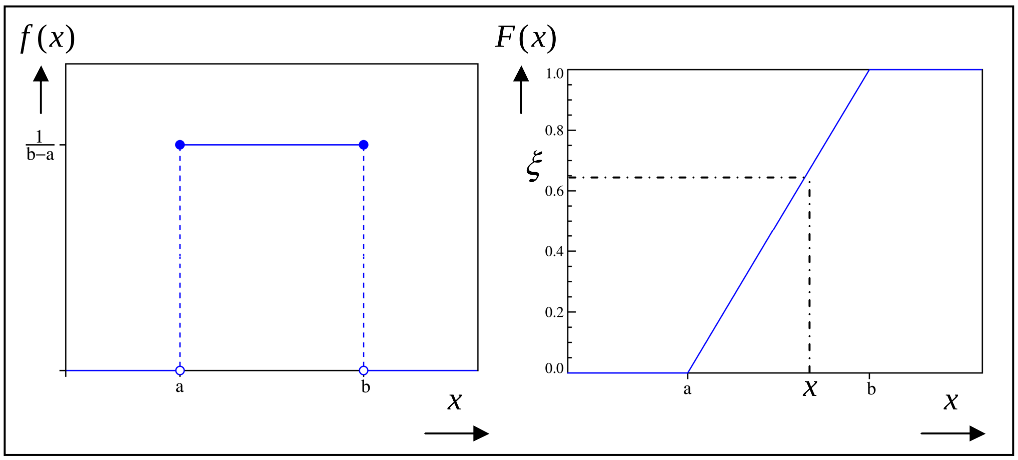


*Figure 10: A uniform probability density function f(x) and its cumulative distribution function F(x) (Wikipedia, 2006).*

The uniform probability density function $f(x)$ on the interval (a,b), and its cumulative distribution function $F(x)$ are as follows (Figure 10):

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a < x < b, \\ 0 & \text{for } x < a \text{ or } x > b, \\ n/d & \text{for } x = a \text{ or } x = b. \end{cases} \qquad F(x) = \begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-a} & \text{for } a \le x < b \\ 1 & \text{for } x \ge b \end{cases} \tag{59}$$

Thus, to uniformly sample the random variable x (Figure 10):

$$x = a + (b - a)\xi \tag{60}$$

where ξ is created by a random number generator uniformly within the interval [0,1].

## 4. Non-linear inversion algorithm flow

The Monte Carlo inversion algorithm is written using a structural programming style. It includes all three basic structures – sequence, selection, and repetition (Deitel and Deitel, 2005). The flow consists of a five main stages, combined into these three structures:

1) Input
    a. Input the following data into algorithm:
        i. The window of seismic data, including the time gross thickness determined by the spectral inversion. The window should be big enough and tapered to avoid the edge affects in the inversion.
        ii. Estimated or assumed probability density functions for the model parameters and the number of layers (type of probability density function, mean, and standard deviation). Here, uniform probability function is used, as it is considered to be the least biased.
        iii. Time thickness from the spectral inversion ($\Delta t_x$).
        iv. Starting acoustic impedance of a layer ($I_y$), which is considered to be constant in the area, estimated from well log data.
        v. Number of trials and iterations ($T$ and $J$).
        vi. Clean sand and shale acoustic impedances ($I_{ss}$ and $I_{sh}$).

2) Random sampling

   a. Randomly sample the number of layers from given input.

   b. Randomly sample the initial model *m'* from the given probability density function (*T* times).

3) Iterations (Figure 11)

   a. For a given number of layers and initial model and input data (*J* times):

      i. Determine Jacobian, and calculate the rank of Jacobian. If rank is less then *3L-1*, this trial is ill-posed; thus, go to the step 2.

      ii. Constrain $\Delta t_x$ and $I_y$ using Lagrange multiplers in to Marquardt-Levenberg method and calculate the perturbation $\Delta m$.

      iii. Perturb the initial model by $\Delta m_{lin}$.

      iv. Compare actual data with the predicted data. If the sum of the squares errors has satisfactory value or if it is the last iteration ($J^{th}$), save the model parameters and Lagrange multipliers values only if the error is the smallest so far with respect to the number of layers and go to the next trial (step 2). Otherwise, go back to the next iteration with the initial mode $m_{lin}$.

4) The number of degrees of freedom

   a. Determine the minimum set of parameters to be used based on function $I_{average} = f(L)$. If $I_{average}$ of the layer of interest become stable increasing *L*, the degrees of freedom are sufficiently high to determine the net-to-gross ratio of the layer of interest.

5) Output

a. Determine a time net-to-gross ratio using the estimated model parameters (starting acoustic impedance ($I_x$) and gradient of the layer of interest ($g_x$)), together with the given clean sand ($I_{ss}$) and shale ($I_{sh}$) acoustic impedance values, using both ray and effective medium theory.

The algorithm for Monte Carlo inversion was written in MATLAB and C programming language. The C code was compiled in a form of dynamic link libraries (DLL), allowing interfacing C functions to MATLAB.

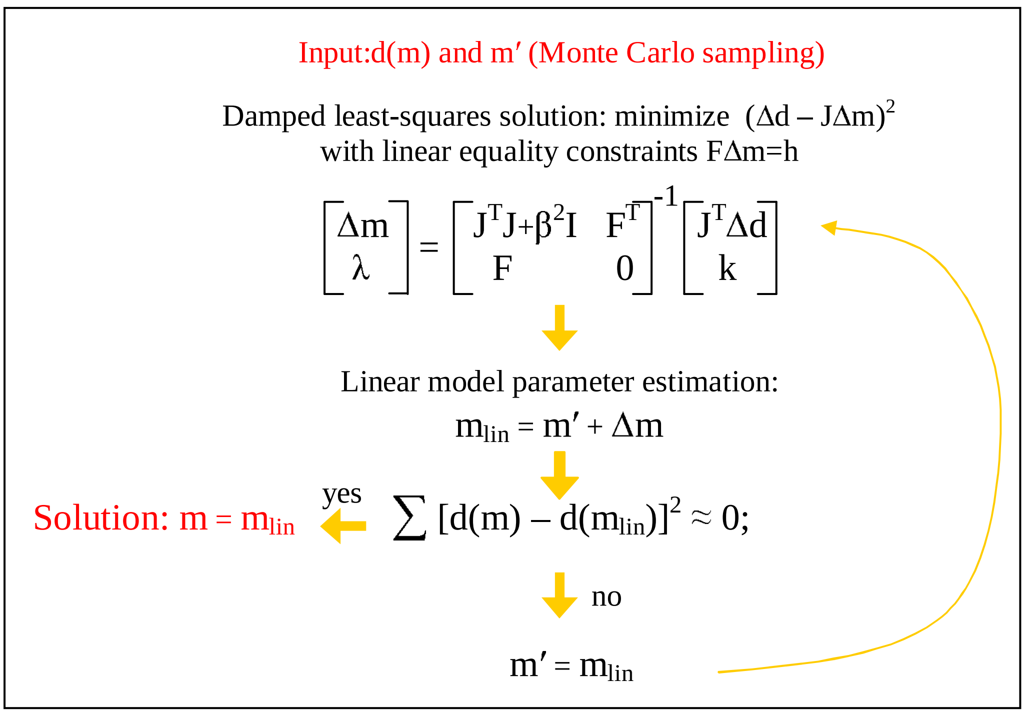



*Figure 11: Non-linear inversion - iterative procedure.*

# 5. Wedge models

To examine how different noise scale with variant travel-time thickness affects the results of the developed algorithm, wedge models are used; geologically, the wedge models are actually pinch-out models. In addition, the wedge models are also used to investigate the effects of the inaccurate constrained values of travel-time thickness and starting acoustic impedance.

To get synthetic data for which the model parameters are going to be estimated, the convolution forward modeling is applied on the actual parameterization of the inversion algorithm, and only a three-layer model is assumed.

## 5.1 The choice of the model parameter values

All parameters except the travel-time thickness of the mid layer (reservoir) are assumed to be constant in the models. In addition, the values for the acoustic impedances of clean sand and shale to be used to calculate net-to-gross ratio using the equation 26 and 28 are considered to satisfy Gardner equation, that is, the empirical chart (Figure 12). Using the chart, the acoustic impedance of the clean sand ($I_{ss}$) is greater of the acoustic impedance of the clean shale ($I_{sh}$) ; that is:

$$I_{ss} = 7.5\ 10^6 [kg/(m^2 s)] \text{ and } I_{sh} = 5.5\ 10^6 [kg/(m^2 s)] \qquad (61)$$

The model parameters used in the inversion are determined arbitrarily. However the values for the starting acoustic impedances for all three layers ($I_1$, $I_2$, $I_3$) are defined to be the values between the clean sand and clean shale values, as all three layers are considered to be the mixtures of the sand and shale. Furthermore, the starting acoustic impedances of the reservoir (layer 2) are greater then the starting acoustic impedance of the underlaying and overlaying layers (layer 1 and 3) in the model (Figure 13), assuming that the reservoir rock contains the higher percentage of the sand relative to the overlaying and underlaying layers. The gradients values ($g_1$,$g_2$, and $g_3$) in the model are taken to be arbitrary, except that they are positive, because in most cases in practice, acoustic impedance increase with time/depth.

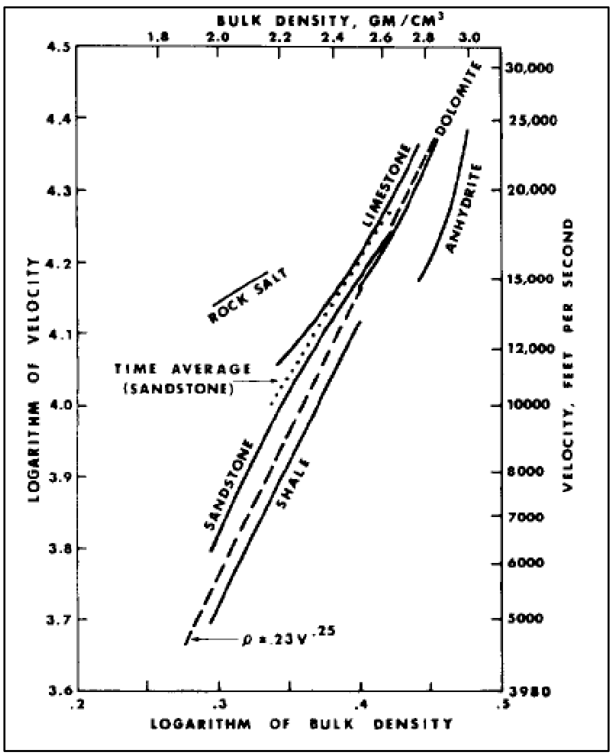



*Figure 12: P-wave velocity and density relationships for different lithology (Gardner et al., 1974).*

Having defined model values, the four wedge models are examined:

1) the model being noise-free

2) the model with 5% noise

3) the model with 10% noise

4) the model with 20% noise

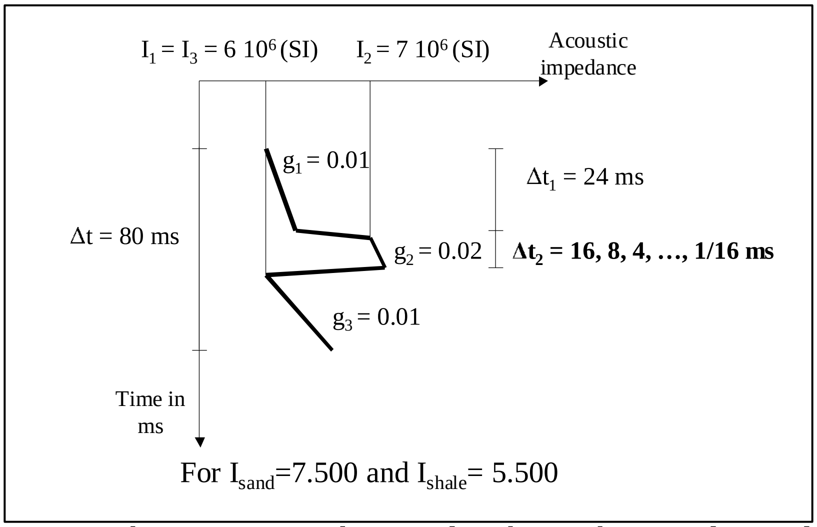



*Figure 13: The parameter values used in the synthetic wedge models.*

First, noise is considered to be random and with the uniform distribution. The maximum amplitude of the random noise is calculated with respect to the signal. One hundred percent of the signal is assumed to be the sum of the absolute values of the maximum and minimum amplitudes of the noise-free data (Figure 14). Because of the tuning effect, these extreme values vary with the thickness. However, the hundred percent of the signal in the algorithm is constant regardless the thickness and corresponds to the response of the model in which the layer thickness is above the tuning travel-time thickness, that is, above the half of the period of the dominant frequency (Figure 14).

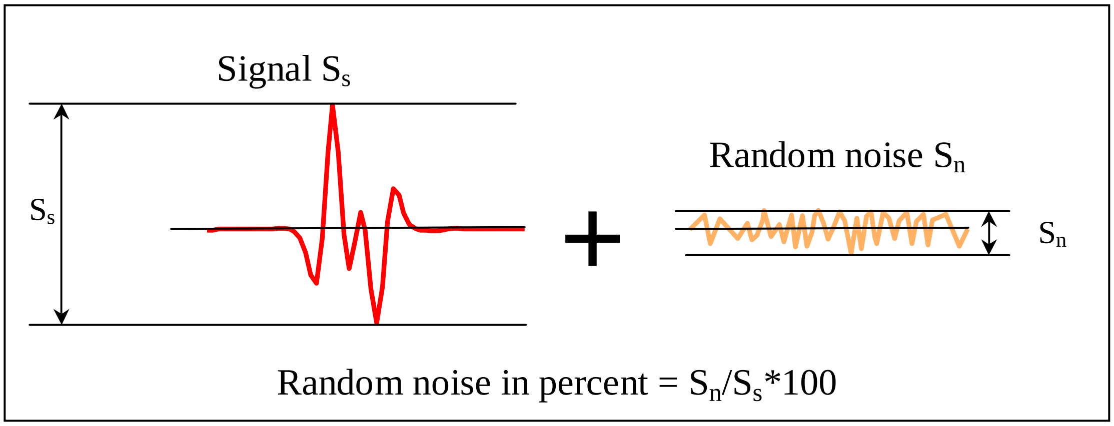



*Figure 14: Definition of the noise added to the synthetic data.*

Having defined the signal and noise in the previous way, the data with 5% noise, for example, are the data with a random uniform noise having maximum (minimum) amplitude of 2.5% (-2.5%) of the signal.

Second, the wedge models response is determined in discrete values of thickness, so that the algorithm has tested thickness above (16 and 8 ms), as well as below (4 ms, 2 ms, 1 ms, ½ ms, 1/4 ms, 1/8 ms, and 1/16 ms) the one-way travel-time tuning thickness. The two-way travel-time tuning thickness for Ricker wavelet, $\Delta t_{2t}$, has been determined by the following equation (Chung and Lawton, 1995):

$$\Delta t_{2t} = \frac{\sqrt{6}}{2\pi f_0} \tag{62}$$

where $f_0$ is the dominant frequency of Ricker's wavelet. Because the dominant frequency of Ricker's wavelet in the synthetic case is considered to be equal to the usual frequency content of the seismic exploration data of 30 Hz, the two-way travel-time separation between the main and side lobe in the case of Ricker wavelet is the following:

$$\Delta t_{2t} = \frac{\sqrt{6}}{2\pi 30} s = 0.013s \tag{63}$$

Therefore, the one-way travel-time tuning thickness is 6.5 ms.

The algorithm used in the wedge models has only 2000 trials of Monte Carlo random sampling. The standard deviation of the sampled uniform probability density function is assumed to be 10 % of the true value of the model parameters. Each trial, that is, an initial model sampled from the uniform probability density function, has gone through sixty iteration of model perturbation to converge towards the neighboring minimum. To create the accurate model data for the one-way travel-time thickness below the tuning effect in a given sampling rate, first the resampling is performed according to the gross thickness, assumed to be determined by spectral inversion. After the resampling the data are resampled back into the original sampling rate. Moreover, during the inversion procedure, that is, perturbation of the model, the resampling has been again applied and resampled back to compare with the response of the model. This approach thus consumes lots of time, especially if the gross/net thickness is very small.

### 5.2 The non-linear inversion results in noise-free environment

The first wedge model is in noise-free environment (Figure 15). Because there is no noise in the data, the solution of the inversion is deterministic. Therefore, the solution of the inversion gives only one result – the estimated "travel-time" net-to-gross ratio based on the ray theory.

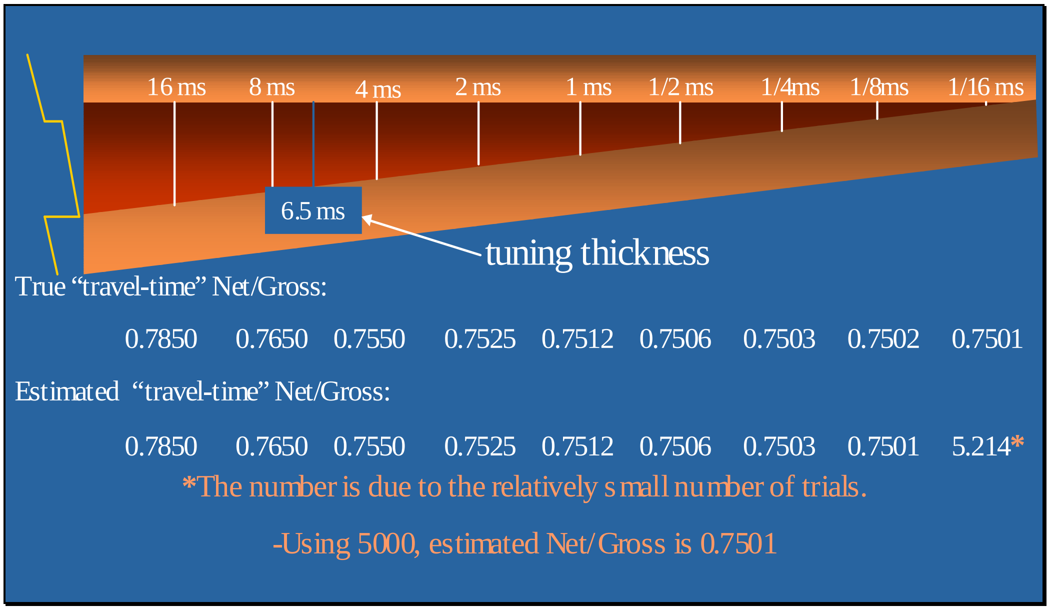



*Figure 15: The noise-free wedge model and the inversion results.*

As it was previously stated above, the non-linear inversion algorithm uses 2000 different initial models to estimate the model parameters, according to the defined probability density functions of the model parameters. Thus, assuming that the gross travel-time thickness of the mid layer is known as well as starting acoustic impedance of the overlaying layer, the estimated travel-time net-to-gross ratios of the wedge model exactly correspond to the true travel-time net-to-gross ratios, that is, the travel-time net-to-gross ratios calculated using the parameters that corresponds to the created synthetic data. However, the estimated net-to-gross for the 1/16-ms one-way travel-time thickness showed an unfeasible result – 5.214. Increasing the number of trials to 5000, the exact solution is achieved, suggesting that the previous solution of the inversion actually converged to the local minimum.

### 5.3 The non-linear inversion results in noisy environment

Although having the same maximum amplitude, the different random noise added into the data gives the different solution of the inversion. Therefore, 200 different random noise sequences (Figure 14) are examined for the same synthetic data corresponding to each travel-time thickness in the wedge model.

The inversion results of the wedge models data that has the added random noise are presented in the statistical form: a histogram with its mean and standard deviation values of the 200 estimated "travel-time" net-to-gross ratios. Thus, each of these estimated ratios corresponds to the different random noise sequence. Figure 16 shows one of the statistical solutions of the inversion. In addition, only feasible results are included into the histogram; in other words, only the estimated travel-time net-to-gross ratios between 0 and 1.

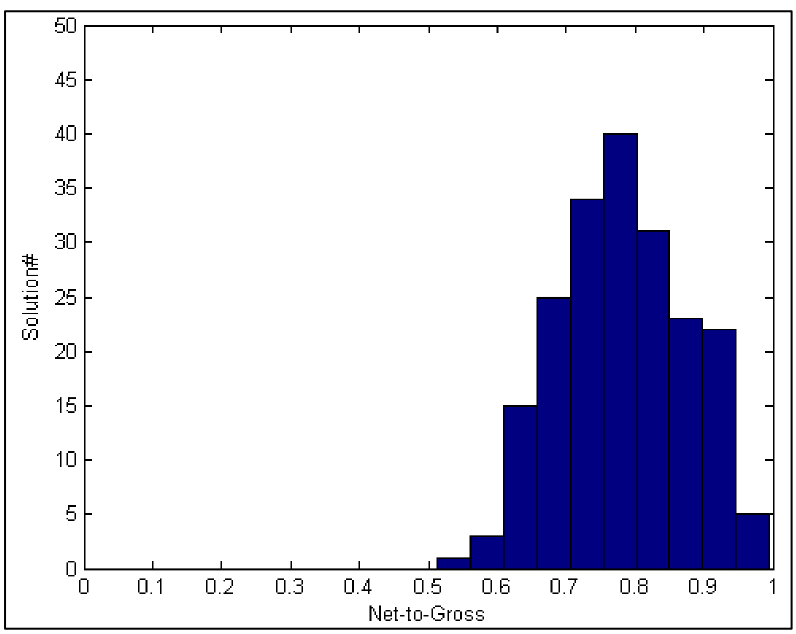
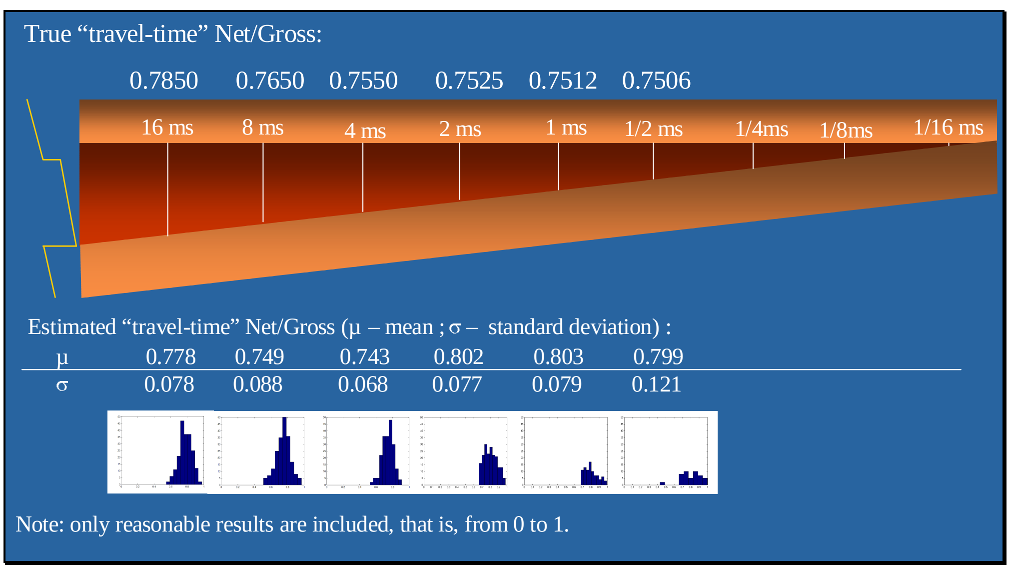


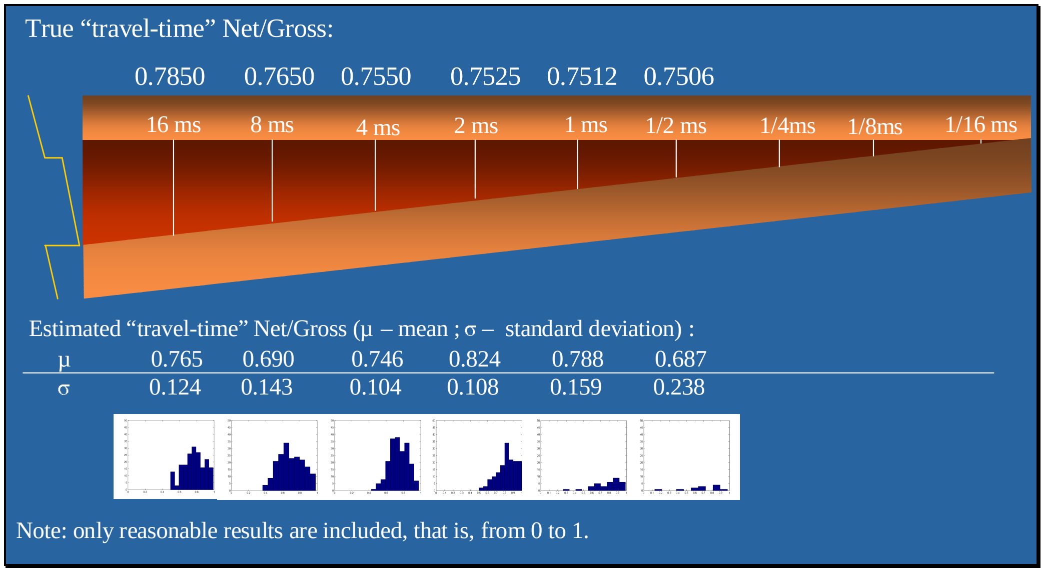

*Figure 16: The inversion results in the form of histogram (200 solutions).*

Again, the three levels of the noise are examined by the inversion algorithm: 5% (Figure 17a), 10% (Figure 17b), and 20 % (Figure 17c). Some inversion results are consistent with the expectation: for each case, the smaller the level of the noise in the model data, the smaller the standard deviation of the solution histogram.

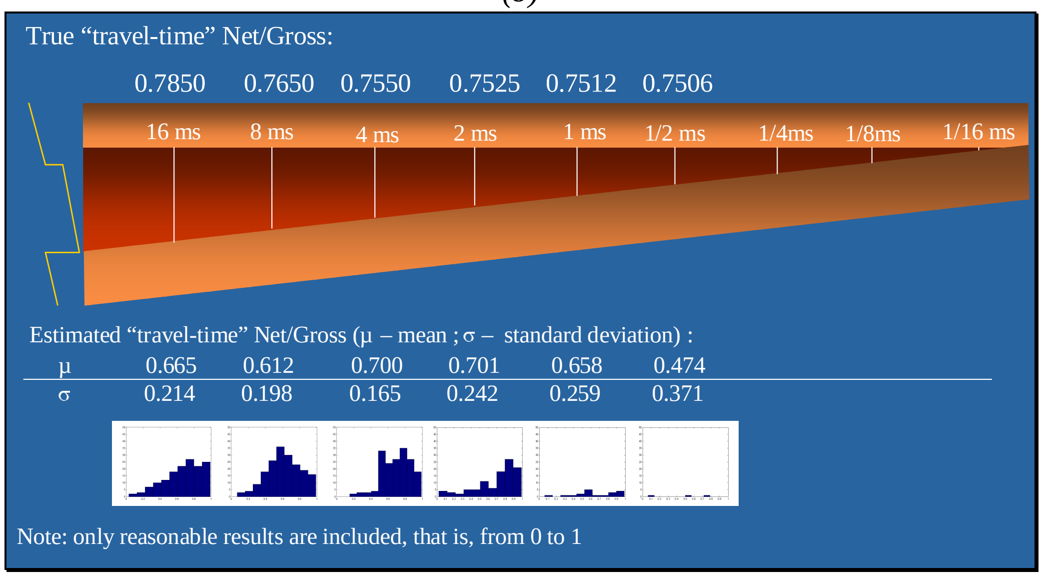
However, one would expect that for each case, the smaller the thickness of the mid layer (reservoir), the bigger the standard deviation of the solution histogram becomes, as the destructive interference of the main and side lobes of the wavelets from the reflections. This behavior of the solutions is not always true: when the travel-time thickness of the reservoir is close to the tuning thickness (6.5 ms), the inversion algorithm gives the solution with smaller standard deviation than the expected. This relation between solutions is due to the constructive interference between the main and side lobe, making the signal stronger.




*(a)*




*(b)*




*(c)*

*Figure 17: The wedge models with (a) 5%,*
*(b) 10%, and (c) 20 % of noise and the inversion results.*

Similar conclusions can be made regarding the mean of the histograms: the smaller the level of the noise in the model data, the smaller the difference between true travel-time net-to-gross and the mean of the solutions. Likewise, the mean estimated for the region of the tuning thickness is sometimes much closer to the true value then the mean estimated for the region of the thicker thickness.

### 5.4 Inaccurate constraints

When the constraints are used in the inversion process, one question naturally arises – What are the effects on the inversion results if the used constraints are inaccurate? In the wedge-model synthetic case, two parameters are constrained by the non-linear inversion algorithm – starting acoustic impedance of the underlying layer, and travel-time thickness of the mid layer. Previously, the inversion of the wedge-models data has been performed assuming that the constraints are accurate.

Before examining the effects of the inaccurate constraints, it is important to analyze the method used to constraint the data. Here, the method of Lagrange multipliers is used, whose mathematical background is explained in Section 3.4 (Non-linear inversion using constraints). In the method, the solution of the non-linear inversion problem corresponds to the optimization-function minimum satisfying the set of the equality equations ( $Fm = k$ ). Each of the equality equations used in the optimization procedure is associated with the Lagrange multiplier ($\lambda$). The Lagrange multiplier is crucial parameter in terms of understanding the effects of the constraint on the solution of the inversion problem.

Implementing the previous discussion in the wedge model case, the starting acoustic impedance of the underlying layer and the travel-time thickness of the mid layer are constrained using accurate and inaccurate values to examine this phenomenon.

### 5.5 Inaccurate constraint – time-travel thickness

To examine the effects of inaccurate travel-time thickness values, which are constrained within the developed algorithm, on the inversion results, the data from the previous wedge models are used; thus, the model parameters values used in the modeling algorithm are the same as in the previous wedge model examples (Section 5.1: The choice of the model parameter values). However the data of only one thickness are investigated; in this case (Figure 18), the data are created modeling the 1-ms travel-time thickness. Thus, the synthetic trace corresponding to the 1-ms travel-time thickness in the wedge model are inverted using the inaccurate constrained thickness: 16, 8, 4, 2, 1/2, 1/4, 1/8 and 1/16 ms. The net-to-gross ratios applying both theories – ray and effective medium – are estimated as well as Lagrange multipliers of the corresponding constraint equality equation that involves one accurate and several inaccurate travel-time thickness.

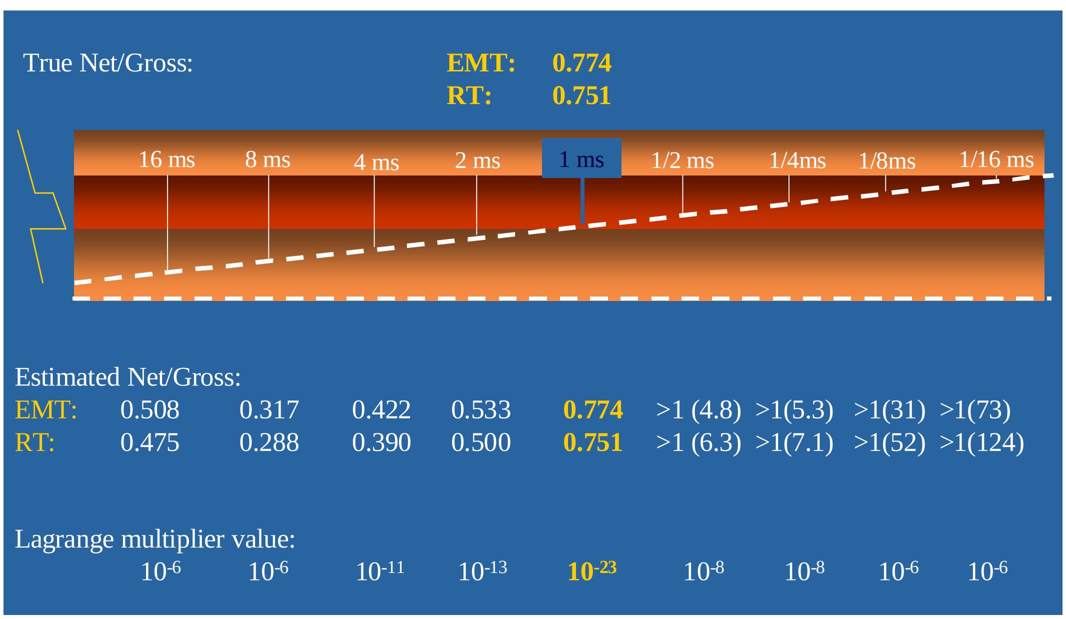



*Figure 18: Inversion results using the inaccurate constraint – travel-time thickness.*

First, regarding the estimated net-to-gross ratios, intuitively, the bigger the discrepancy between the accurate and used thickness, the biggest the error between the true and estimated net-to-gross ratio. Second, the Lagrange multiplier value associated with the accurate travel-time thickness is closest to zero ($10^{-23}$) comparing with those of the inaccurate travel-time thickness.

This relation between the Lagrange multipliers of accurate and inaccurate constraint equality equations suggests, keeping in mind the previous discussion about the physical meaning of the multipliers (Section 3.4 Non-linear inversion including constraints), that the travel-time thickness does not have to be constrained to have the unique solution. Because the travel-time thickness is actually estimated using the same set of data by spectral inversion and thus does not represent *a priori* information, it is not surprise that this parameter does not have to be constrained in the inversion algorithm.

However, in the real example, the spectral inversion determined travel-time thickness is going to be constrained and examined, as it is a mean of 1) biasing the inversion in directions found to be desirable for interpretation purposes, 2) reducing the dimension of initial-model space, and 3) determining the resample rate in the inversion algorithm.

In addition, the net-to-gross ratios estimated are plotted as a function of the constrained travel-time thickness (Figure 19). Clearly, the relations between these two quantities are non-linear.

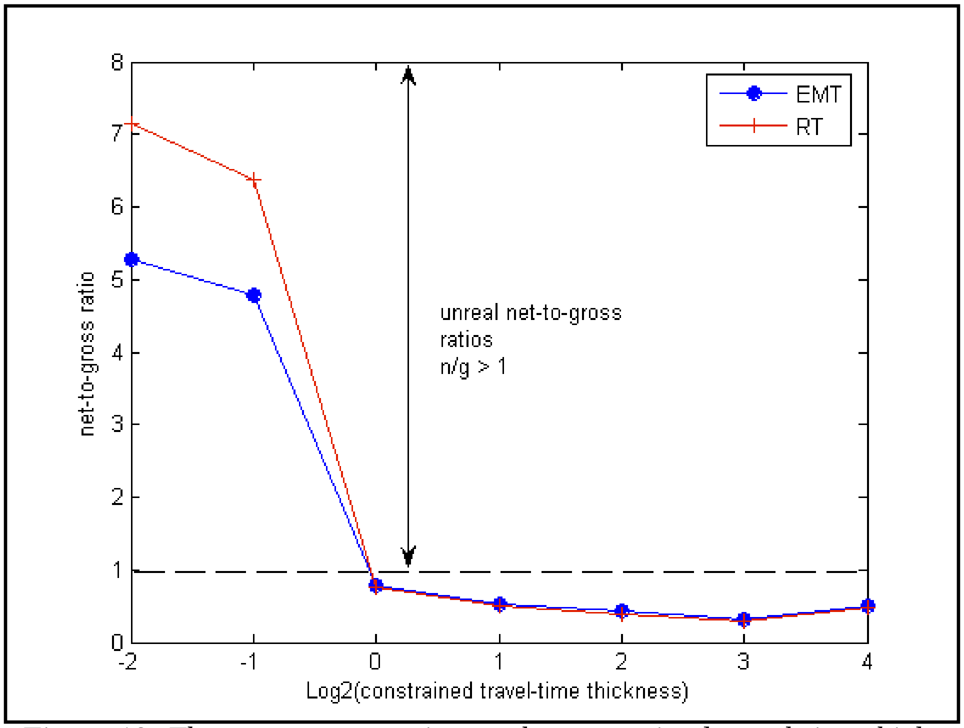



*Figure 19: The net-to-gross ratios vs. the constrained travel-time thickness.*

### 5.6 Inaccurate constraint - starting acoustic impedance

The same synthetic data are used to examine the effects of the inaccurately constrained starting acoustic impedance of the overlying layer. However, in this case, the assumption is that the travel-time thickness is known correctly, whereas the starting acoustic impedance is constrained using one accurate and several inaccurate values within the range from 4 to 8 $10^{-6}$ SI. The net-to-gross ratios as well as Lagrange multipliers values corresponding to the given constraints are estimated and calculated (Figure 20), applying both ray and effective medium theories.

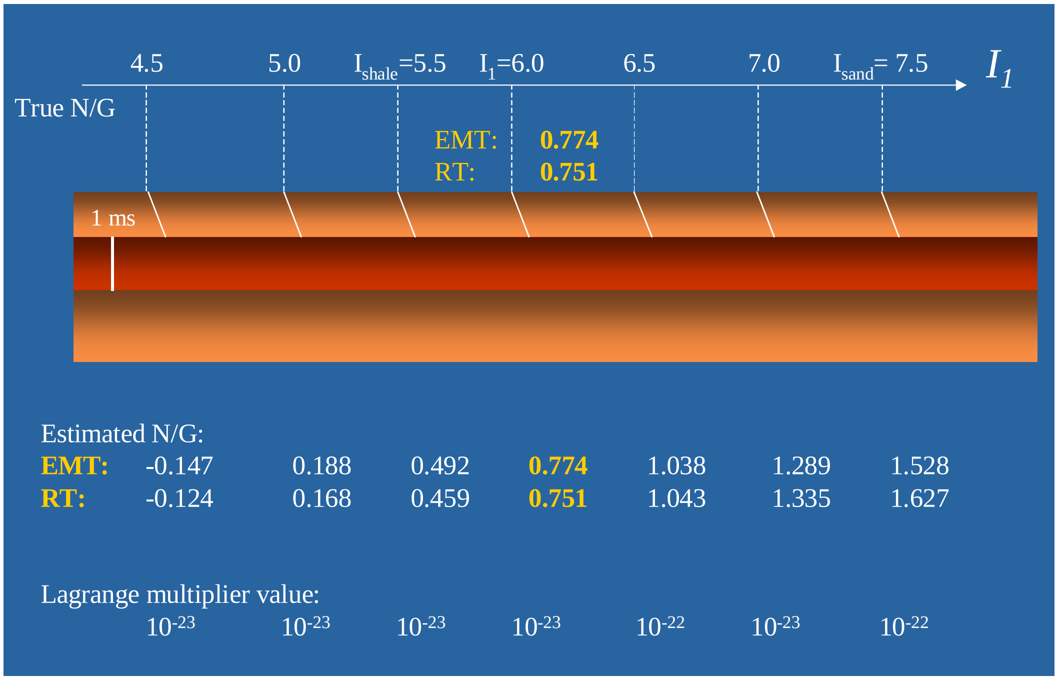



*Figure 20: The inversion results using inaccurate constraint – starting acoustic impedance.*

Again, here the bigger the error of constrained value, the bigger the error in estimated results. However, here in a case of ray theory bound, the relation between the constrained value and estimated net-to-gross ratio are linearly dependent, whereas the relation, in a case of effective medium theory bound, is still non-linear but very close to linear (Figure 21).

Speaking of the estimated Lagrange multipliers, regardless of the constrained value, whether it is accurate (6.5) or not, their value are the same! These constant multipliers suggest that all constraints have the equal "force" to change the minimum of the objective function. In addition, the solution of the inversion is non-unique, as the objective function does not have a single global miminum. Therefore, such a parameter has to be constrained with *a priori* information to have a unique solution. Thus, in the real example (chapter 4. Real example), a starting acoustic impedance of one of the layers is constrained using the value determined from the different technique – well logging.

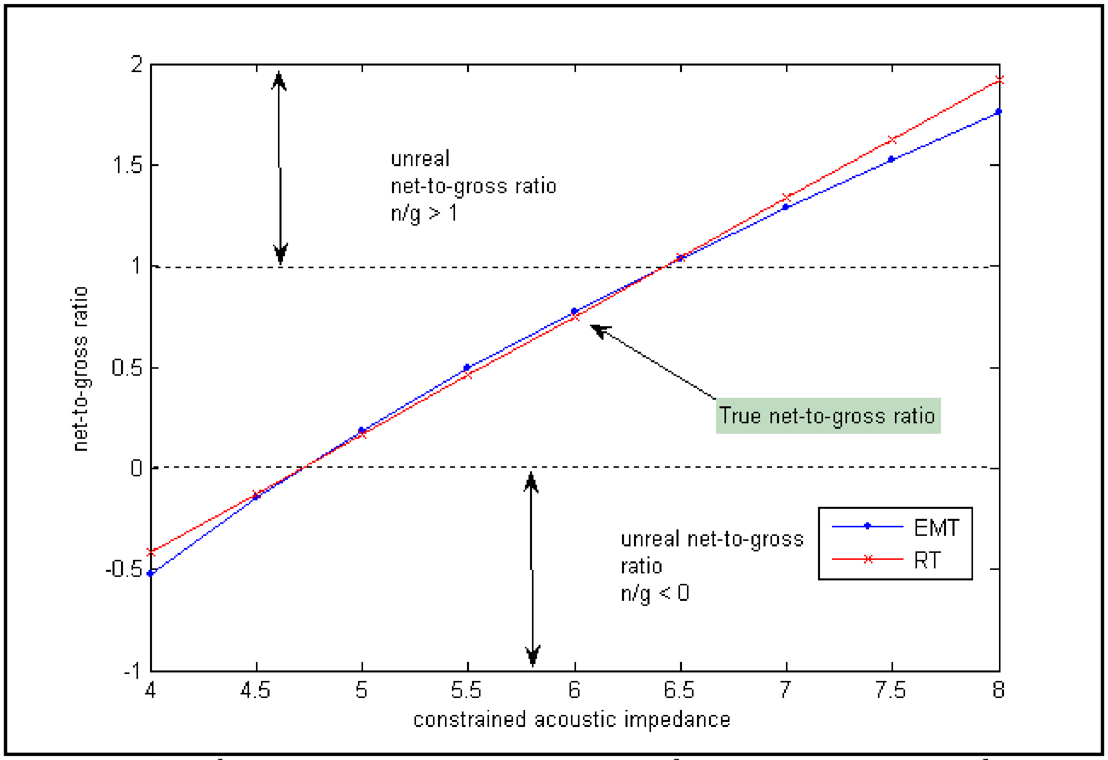



*Figure 21: The net-to-gross ratio vs. constrained starting acoustic impedance.*

## 6. Synthetic example – Multi-layer case

In the previous example, wedge models, there were three assumptions limiting the implementation of such an algorithm into the real case:

1) The number of layers is known (though it could be estimated from spectral inversion).
2) Only three layers are used.
3) The synthetic data were generated using the actual parameterization (unrealistic impedance profile).

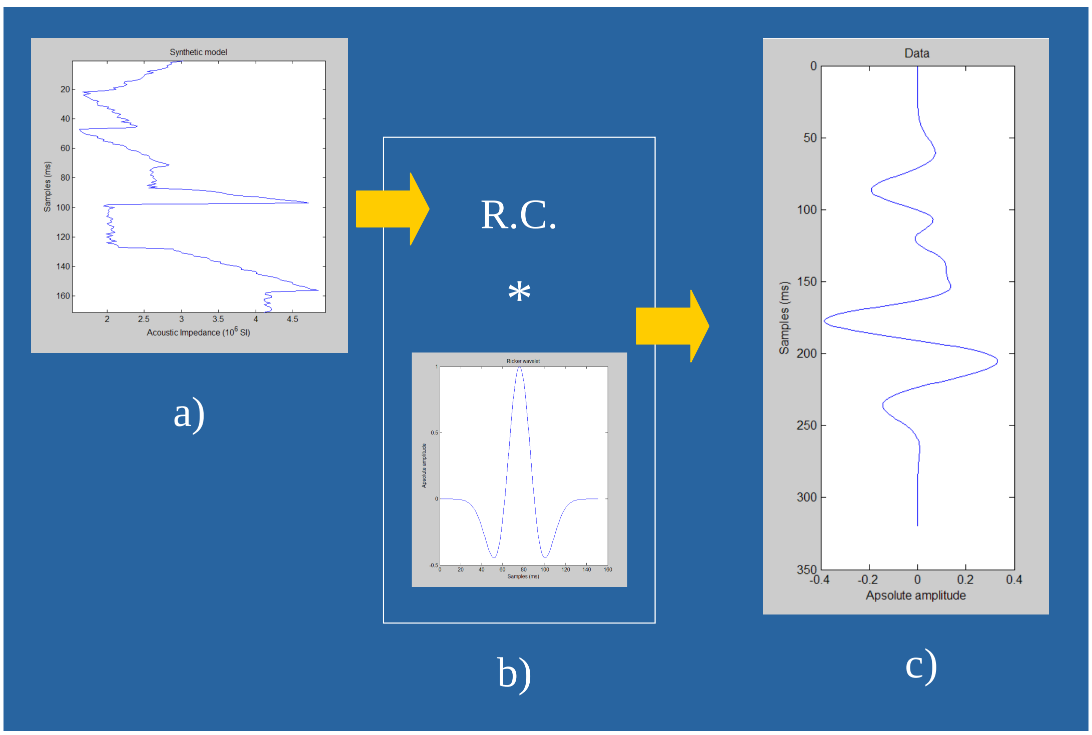



*Figure 22: Generating synthetic data to be used in the inversion.*

Therefore, another synthetic has been investigated. In the synthetic case, all three previous assumptions are excluded: that is, the number of layers is considered to be unknown and arbitrary, and the data are more realistic by adding the uniform random function on the acoustic impedance profile before creating the reflection coefficient for the synthetic data: on Figure 22, it is shown how the synthetic data has been created: First, the arbitrary acoustic impedance profile has been created (Figure 22a); then, the reflection coefficient profile is derived form the acoustic impedance profile, and finally, by convolution of Ricker wavelet and the reflection coefficient profile (Figure 22b), the data have been generated (Figure 22c).

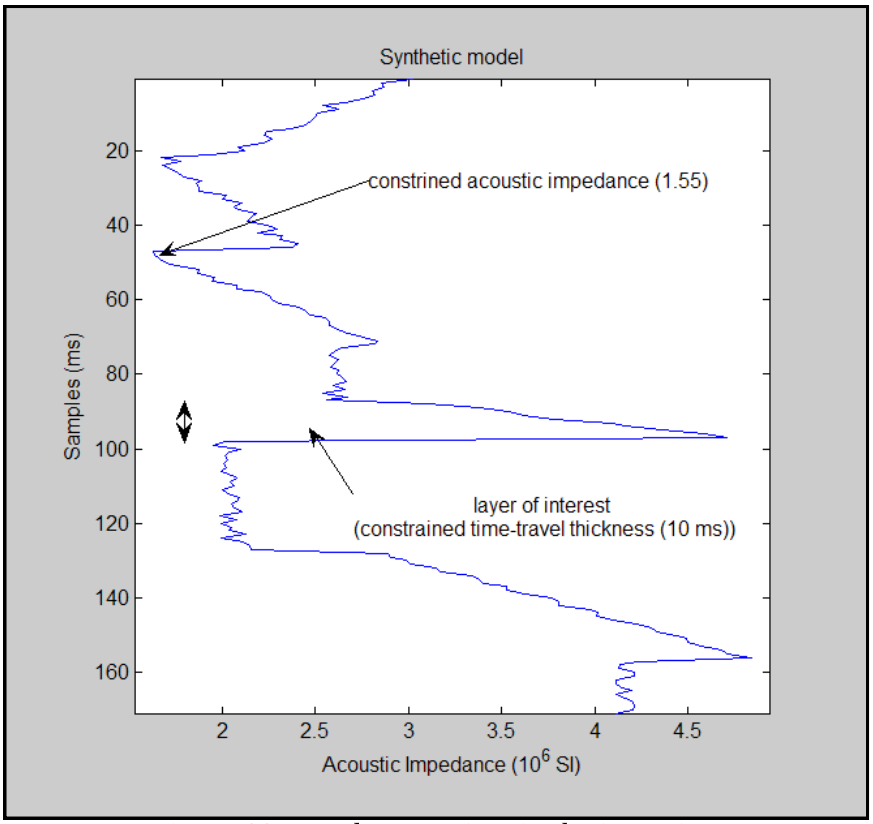


*Figure 23: Used constraints in the inversion.*

Now, in the inversion procedure, as in the previous synthetic case, two parameters have been constrained: a two-way travel-time thickness and starting acoustic impedance of one of the layers. The assumption is that the former is known from the spectral inversion and the latter is known from the well log interpretation. Their values are respectively, 10 [ms] and 1.55 $10^6$ [kg/m3 m/s] in this case (Figure 23).

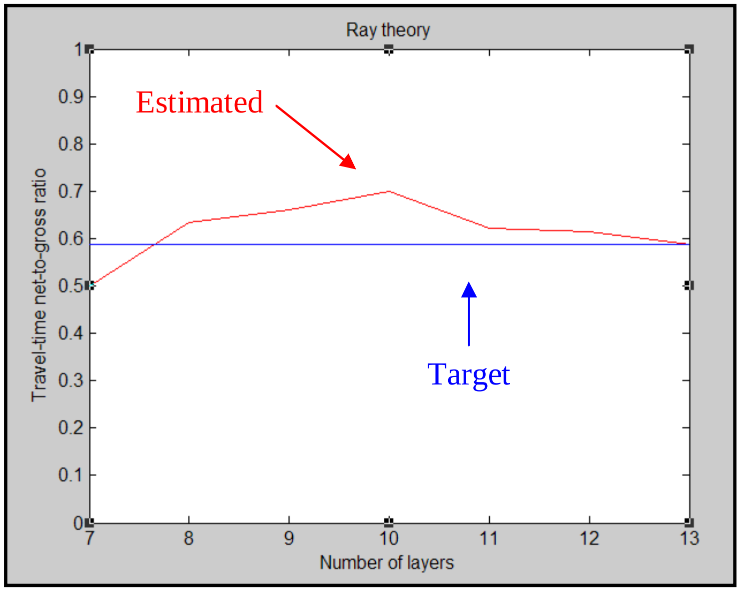


*Figure 24: Ray theory "travel-time" net-to-gross ratio as a function of a number of layers.*

The inversion algorithm has investigated the affect of the different number of layers. There is only one deterministic solution of net-to-gross ratio per the corresponding number of layers for both theories – ray and effective medium (Figure 24 and 25, respectively). These solutions of the inversion correspond to the global minimum solution of objective function, that is, the smallest error between the synthetic data and estimated data for each number of layers. The values of the objective function with the respect of the number of layers used in the inversion are plotted on Figure 26: It is clear that the more degrees of freedom used in the inversion, the less error between the actual and predicted data. In addition, the values of the Lagrange multipliers associated with the used constraints are shown on Figure 27.

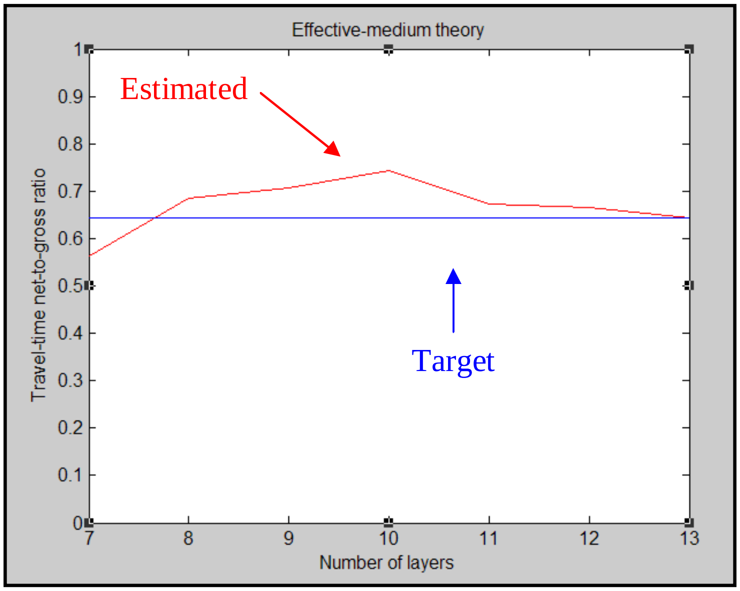



*Figure 25: Effective medium theory "travel-time" net-to-gross ratio as a function of a number of layers.*

Overall, $5 \cdot 10^7$ trials were used to investigate different initial models with different number of layers, assuming that the values are within the two standard deviations from the mean of the uniform distribution, which is the input for the random sampling.

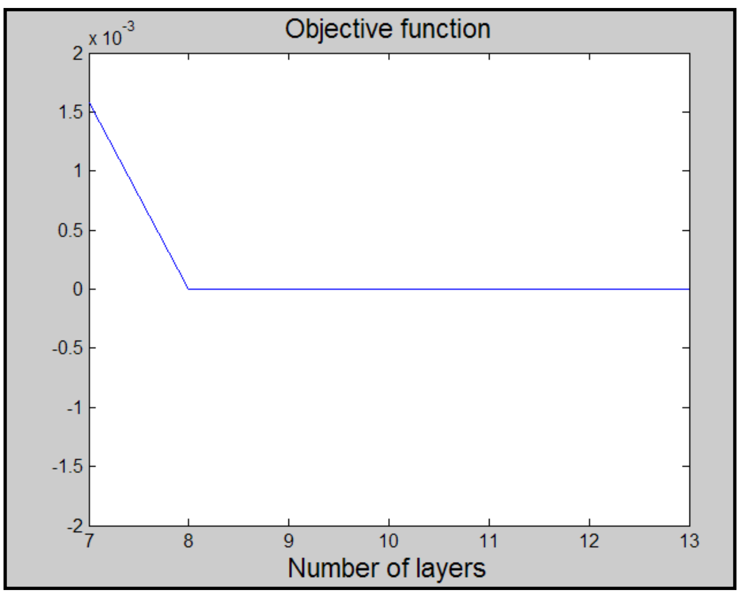



*Figure 26: Objective function as a function of a number of layers.*

According to the values of net-to-gross ratio with the respect to the number of layers for both theories, ray and effective medium (Figure 24 and 25, respectively), the following can be concluded: by increasing the number of layers and thus degrees of freedom, the estimated net-to-gross ratio approaches to the “true travel-time” net-to-gross ratio. In this case the model that has thirteen layers gives the very good estimation of the net-to-gross ratio (Figure 34). This behavior actually means that the average of acoustic impedance using the thirteen layers are the same as the average of the target model.

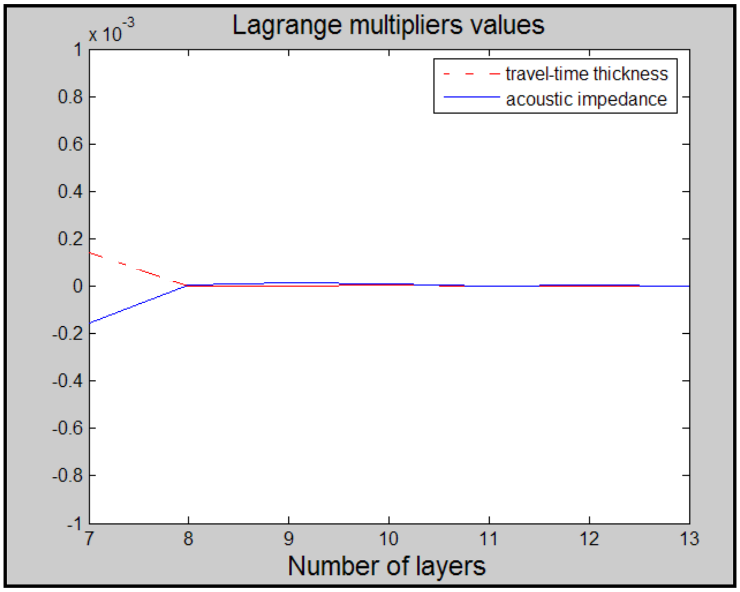



*Figure 27: Lagrange multipliers of two constrained parameters as a function of a number of layers.*

This conclusion has been already seen in the inverse problems: the higher the degrees of freedom, the better match between the actual and estimated model. For example, it can be seen in designing matching filter in so-called Wiener least-square filtering. The longer the length of the filter, the better matching between the input and desired output (Robinson and Treitel, 2001).

On the other hand, it is interesting to mention that the best match for the interpretation of the gradients gives the model that uses the only eight layers (Figure 29). According to the results, objective function (Figure 26) and Lagrange multipliers that correspond to the constraints (Figure 27) show abrupt drop in their value, at that point where the solution should be used for geological interpretation.

The estimated models that correspond to the global minima for each of the number of layers are show bellow (Figure 28-34).

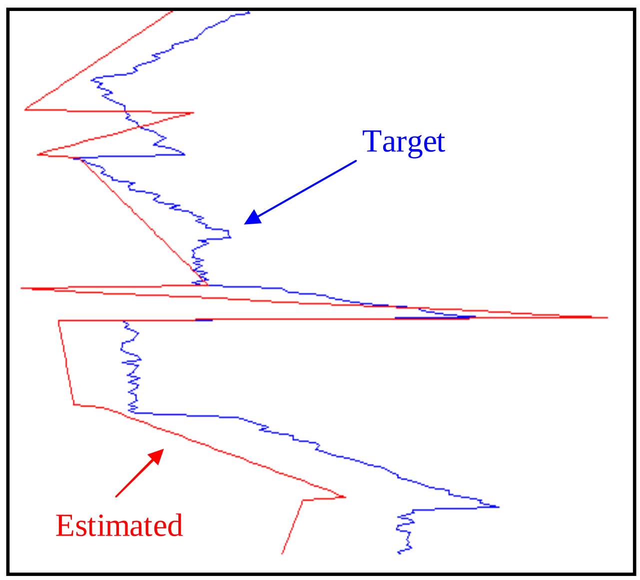



*Figure 28: Inversion solution using seven layers.*

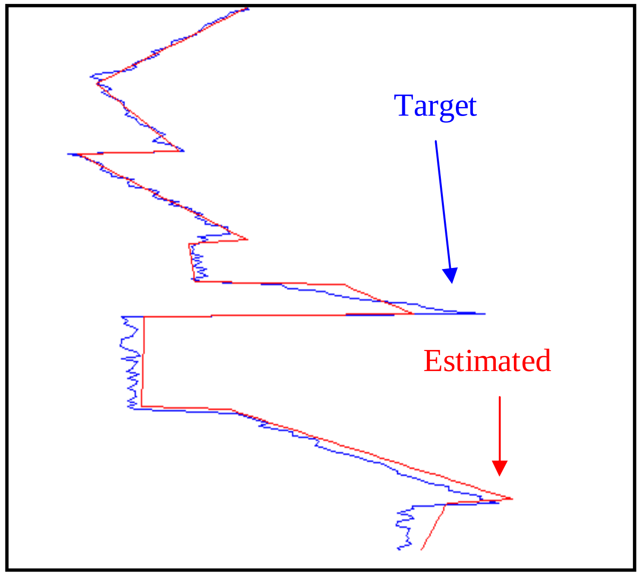



*Figure 29: Inversion solution using eight layers.*

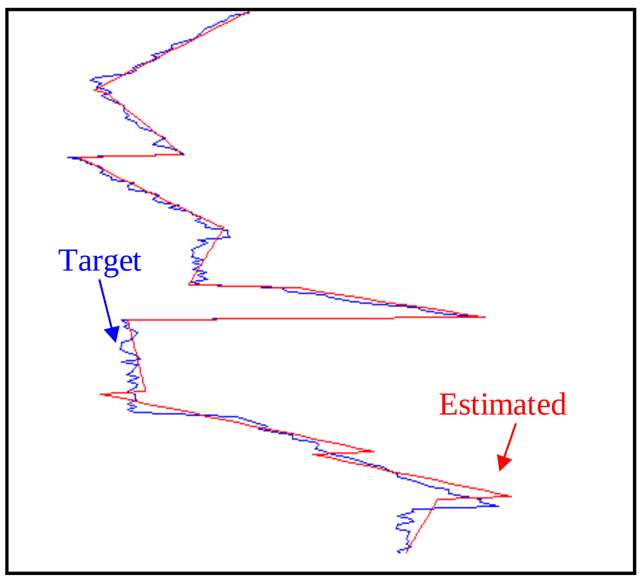



*Figure 30: Inversion solution using nine layers.*

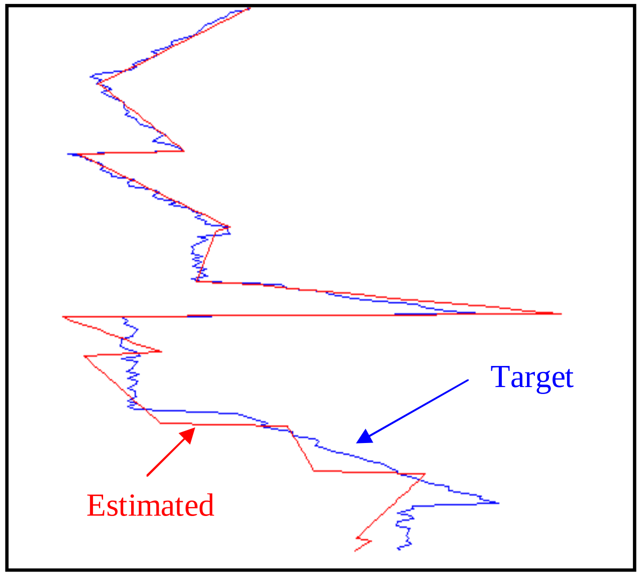



*Figure 31: Inversion solution using ten layers.*

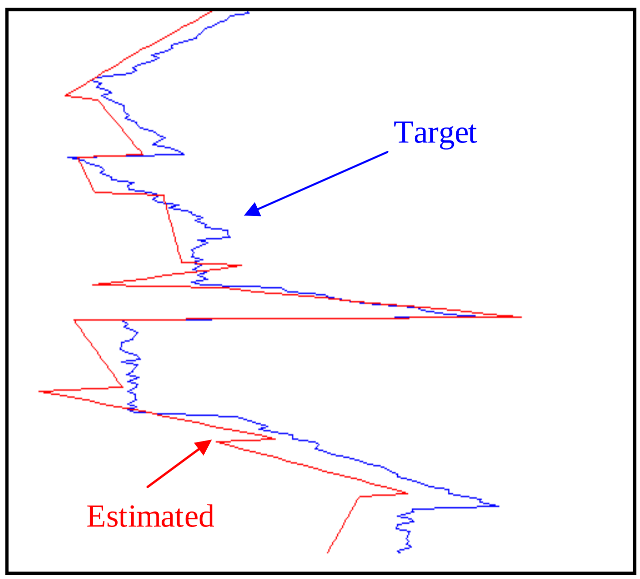



*Figure 32: Inversion solution using eleven layers.*

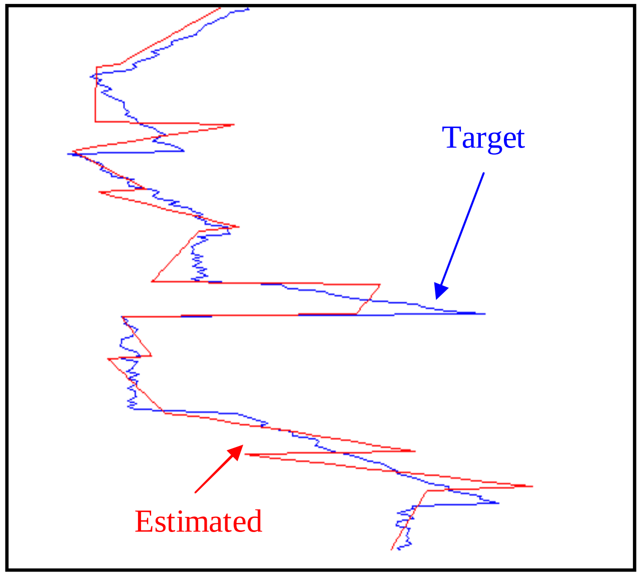



*Figure 33: Inversion solution using twelve layers.*

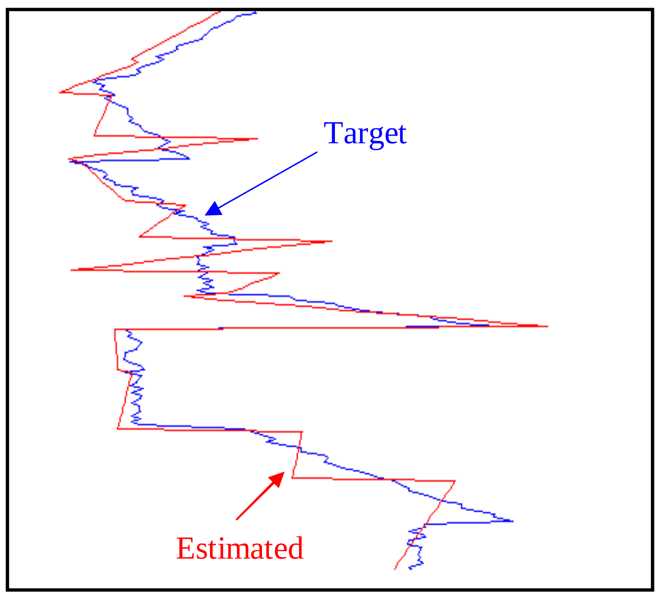



*Figure 34: Inversion solution using thirteen layers.*

## 7. Real example – South Timbalier field

The South Timbalier field is located south from Louisiana in the Gulf of Mexico (Figure 35). The available data from the field in the research are the post-migration stack volume as well as geophysical well log data. Although there is available spectral inversion volume for the area, it has to be discarded from the research as it was derived using inaccurate synthetic ties from the previous study. Therefore, the algorithm is going to use the constrained travel-time thickness from well logs rather then from the spectral inversion results to estimate the feasibility of the developed algorithm.

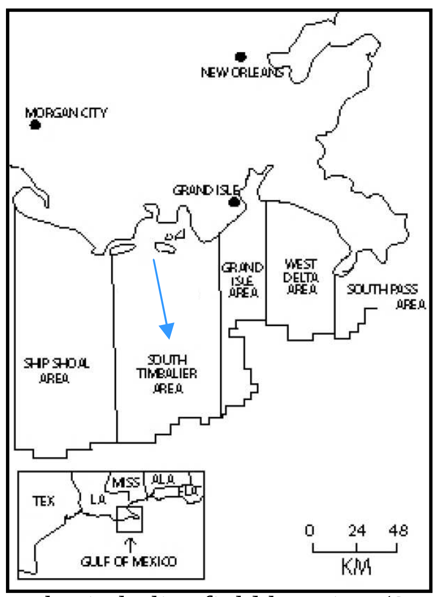



*Figure 35: South Timbalier field location (Stude, 1978).*

The geological environment of South Timbalier field is typical for the Gulf of Mexico: sedimentary basins with various salt domes and structures (Stude, 1978).

Because only one well at the location has the acoustic log data acquired, allowing the accurate estimation of the seismic velocity of the formations, the research was limited only to that well (Figure 36). In addition, because the density logs were not available in the well, the acoustic impedance was not directly calculated but estimated from the given velocity log. The estimation from the velocity should give sufficient accuracy for the application, assuming the constant density for the given data window.

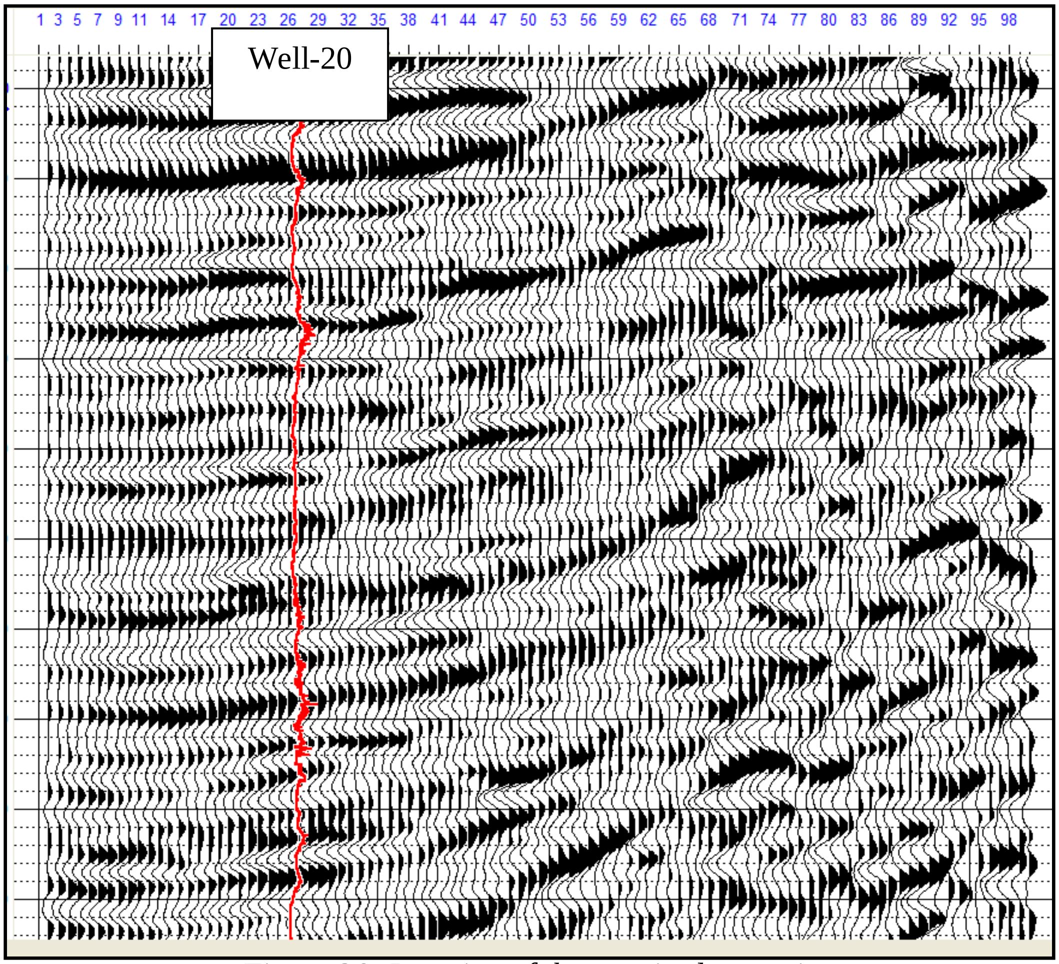



*Figure 36: Location of the acquired acoustic log on the migrated seismic section – Inline 607.*

As it could be seen on Figure 36, the well 20 is drilled through a sedimentary basin; thus, by identifying the shale-sand sequence on the log, it allows us to implement the developed algorithm for a given sequence. Such a sequence has been identified (Figure 37 – red dashed arrow). According to log data (Figure 37), higher impedance might suggest that sequence should have more sand content then shale. Within the sequence, the target layer has been defined (Figure 37 – black solid arrow). The target layer is very thin; the two-way travel time thickness of the layer is 8 ms.

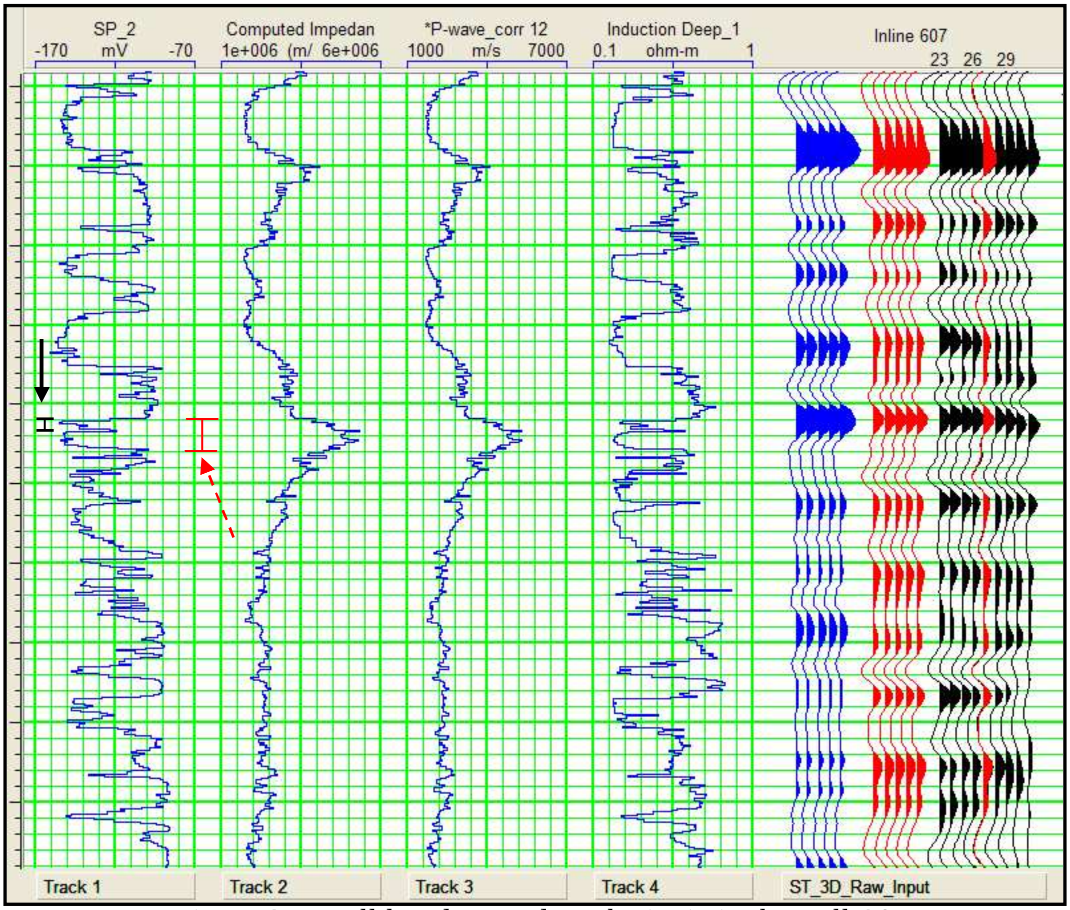



*Figure 37: Well log data and synthetic tie in the well-20.*

To efficiently apply the developed algorithm, it is crucial to have a good correlation between the acoustic impedance profile and some net-to-gross ratio indicator, such as gamma-ray or electrical self-potential (SP) well logs in the shale-sand sequence of the interest. For the given well, only SP log is available, and there is a correlation between the acoustic impedance (track 2) and self-potential log (track 1) in the area of interest, making the algorithm applicable (Figure 37).

To test the feasibility and application of the research algorithm, the results of net-to-gross ratio derived from the trace located nearest to the well are compared with the results of net-to-gross ratio derived directly from the well log data. As a matter of fact, the constrained data, that is, starting acoustic impedance of one of the layers and travel-time thickness of the layer of interest, are used from the same well log data.

### 7.1 Estimating net-to-gross from the well logs

As it was stated before, the electric self-potential (SP) log can be used to determine a net-to-gross ratio of a layer. The net-to-gross ratio is defined by the following equation:

$$Net/Gross_z = \frac{h_{ss}}{h} = \frac{SP - SSP_{sh}}{SSP_{ss} - SSP_{sh}} \qquad (64)$$

where $SP$ is electrical self-potential of the layer of interest, $SSP_{ss}$ is a electrical self-potential of clean send, and $SSP_{sh}$ is a electrical self-potential of clean shale. Using the previous equation, the target layer in the research has the net-to-gross ratio of 0.85 (Figure 37 – black arrow). Because electric self-potential is measured in depth, such a determination of the net-to-gross should represent "true" net-to-gross ratio.

In addition to estimating net-to-gross using the SP curve, the acoustic impedance can be used to estimate travel-time net-to-gross ratio (Section 2.4 Calculating reservoir travel-time net-to-gross ratio). Therefore, from equations 26 and 28, using ray and effective medium theory respectively:

$$Net/Gross_t^{rt} = \frac{I_{mean} - I_{sh}}{I_{ss} - I_{sh}} \text{ and, } Net/Gross_t^{emt} = \frac{I_{ss}\left(I_{mean}^2 - I_{sh}^2\right)}{I_{mean}\left(I_{ss}^2 - I_{sh}^2\right)}$$

the calculated net-to-gross for the target layer is 0.87 and 0.89, respectively, which is very close to the "true", derived from SP curve – 0.85.

### 7.2 Estimating net-to-gross from the non-linear seismic inversion

Before the data are included into the developed algorithm to estimate travel-time net-to-gross ratio, they have to be prepared with the following steps:

1) Perform the synthetic ties on the data in order to scale the amplitudes of the trace to their absolute values. Here, STRATA (Hampson-Russell) software has been used for the step. In this case, the correlation between the synthetic and original trace has achieved the value of 0.82 (Figure 38a).
2) Determine the window of the trace that includes the layer of interest and the anchor layer with the known value of starting acoustic impedance throughout the area of interest. The good choice for the anchor layer could be some shale sequence with a constant value throughout the field. It should be kept in mind that the layers should not be affected by the edge effect, that is, the length of the used window should be sufficiently large (Figure 38b).
3) Apply the tapers on the edges of the trace window so that the edge effects are reduced (Figure 38c).
4) Determine the values of the data to be constrained, that is, starting acoustic impedance of the anchor layer and two-way travel-time thickness of the layer of interest (Figure 38d).

5) Determine the acoustic impedance of the clean sand and clean shale. These should be the minimum and maximum acoustic impedance values in the shale and sand sequences, respectively for this case. Using the well logs, the determined values for the clean shale and the clean sand are 1.6 and 5.4 $10^6$ SI units, respectively.

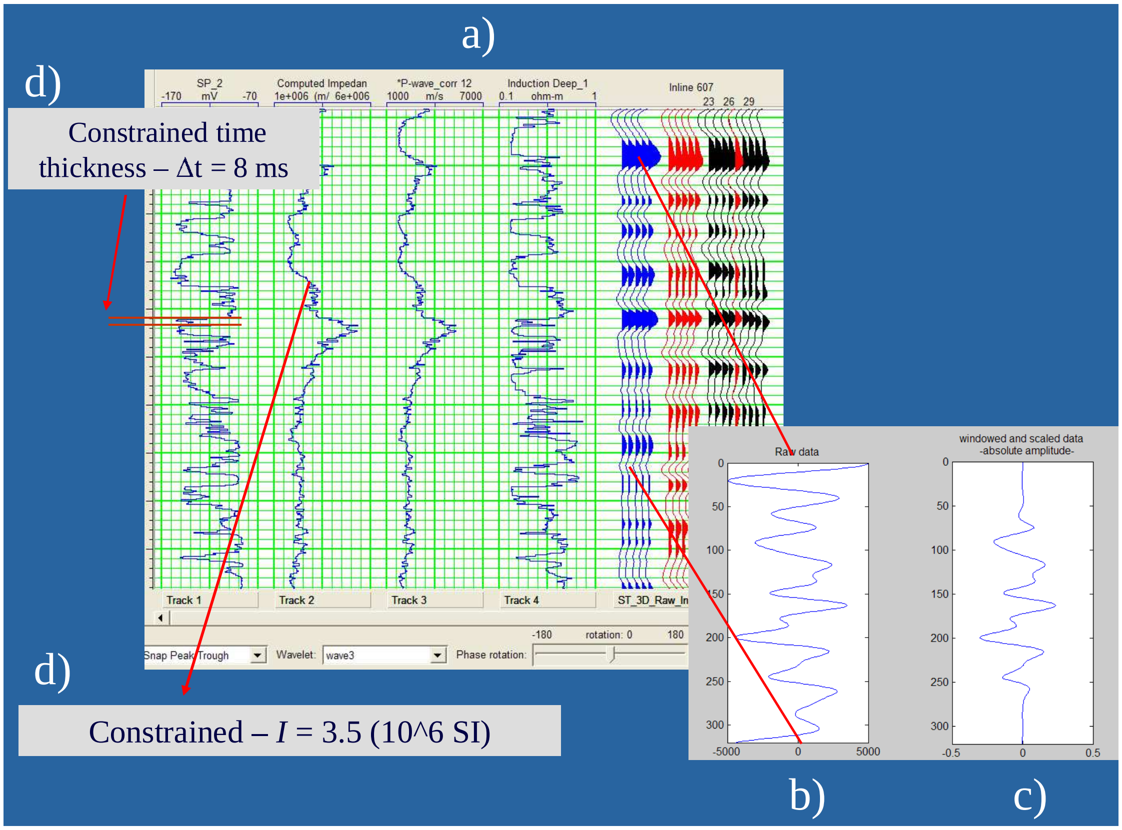



*Figure 38: Data preparation.*

Now that data has been prepared the algorithm can be used to estimate the acoustic impedance profile of the layer of interest and thus “travel-time” net-to-gross ratio, assuming different number of layers in the model.

As it was previously stated in the multi-layer synthetic case, random sampling technique has been used to investigate different initial models. The number of trials that has been used here is about 40 million, roughly 5 million per one thickness. Therefore, for each thickness there is one global minimum solution, which has been used for determining travel-time net-to-gross ratio.

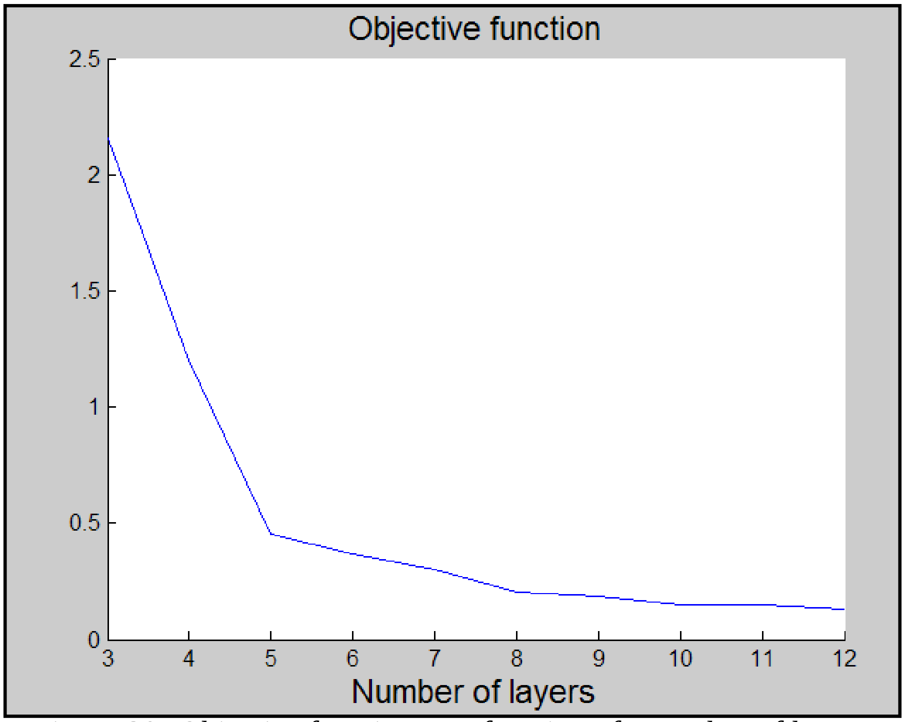



*Figure 39: Objective function as a function of a number of layers.*

Here the observations are the same as in the synthetic multi-layer case – First, the more layers in the model, that is, the higher degrees of freedom in the inversion, the smaller the objective function (Figure 39). Second, the higher the degrees of freedom, the more closely are the estimated net-to-gross ratio from seismic data to the estimated net-to-gross ratio from the well log data (Figure 40 and 41). In other words, the estimated acoustic impedance average of the layer of interest is closer to the average of acoustic impedance determined from the well log data. However, what has been observed is that with the higher degrees of freedom, the calculated Jacobian is much often singular for a given initial model; thus, it is more difficult to achieve well-posed solution. Moreover, the computational time increases by increasing the number of layers, that is, model vector dimension.

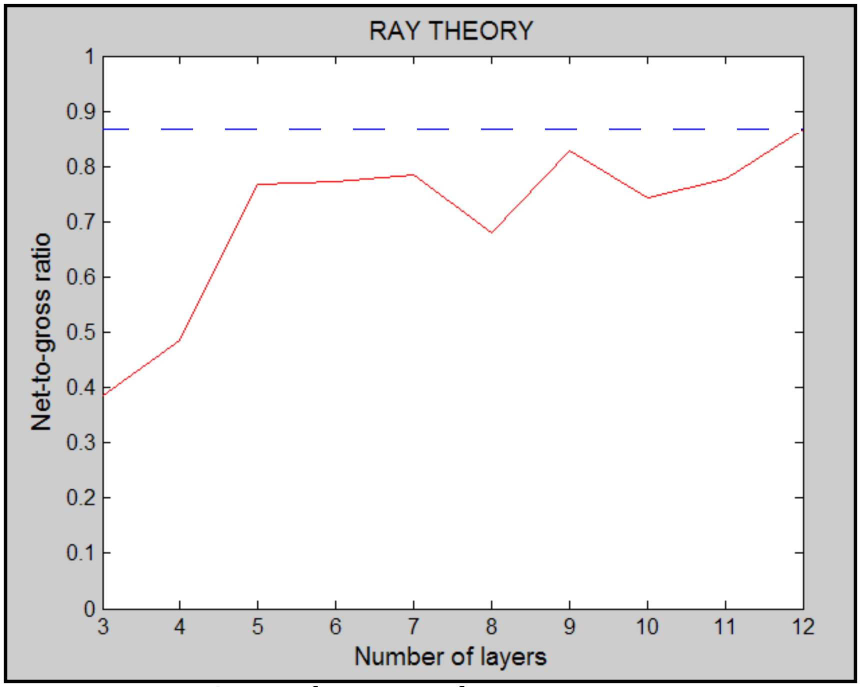



*Figure 40: Ray theory travel-time net-to-gross ratio as a function of a number of layers.*

In addition, the interesting question stayed unsolved - could we have known at the beginning what is the sufficiently number of layers to be used in the inversion?

First, the answer to the previous question would have given the spectral inversion technique, but as it is not accurate, we could not use it. Second, we can always do the same scanning through different number of layers and see when the value of the net-to-gross ratio (that is, acoustic impedance average) of the layer of interest is going to be stable, and than use the result from the highest degrees of freedom. This approach is feasible only if the computational time is not the issue.

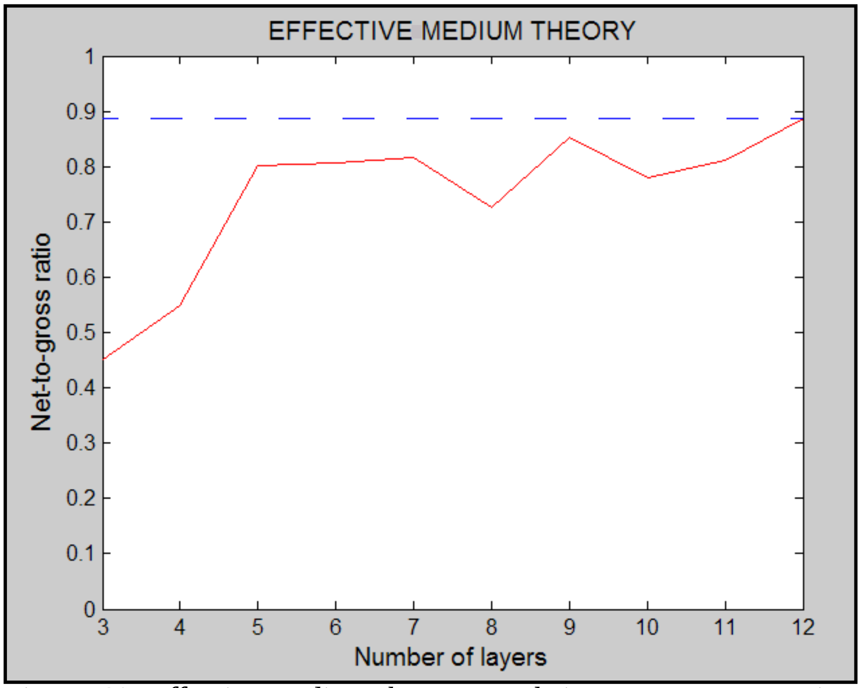



*Figure 41: Effective medium theory travel-time net-to-gross ratio as a function of a number of layers.*

The results of the global minimum solution for even number of layers are shown below (Figure 42, 43, 44, 45, and 46). The odd numbers of layers are skipped just for simplicity. It should be stated that with the high degrees of freedom the estimated acoustic impedance gradients of the layers are not interpretable. However, the general trend of the estimated acoustic impedance gradients by the inversion has good match with the regional trend of the well-measured acoustic impedance gradients (Figure 46).

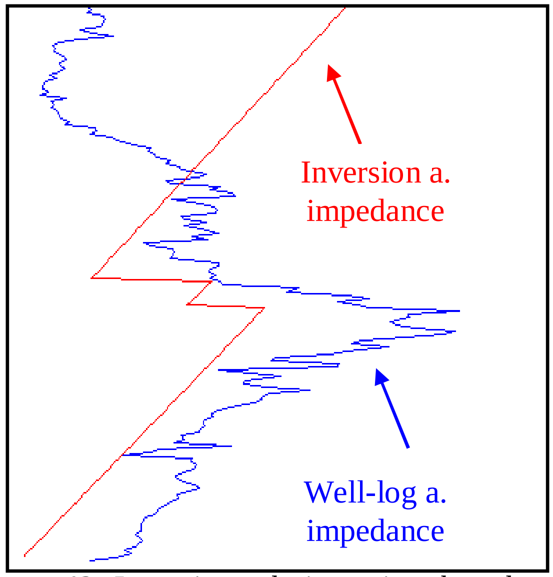


*Figure 42: Inversion solution using three layers.*

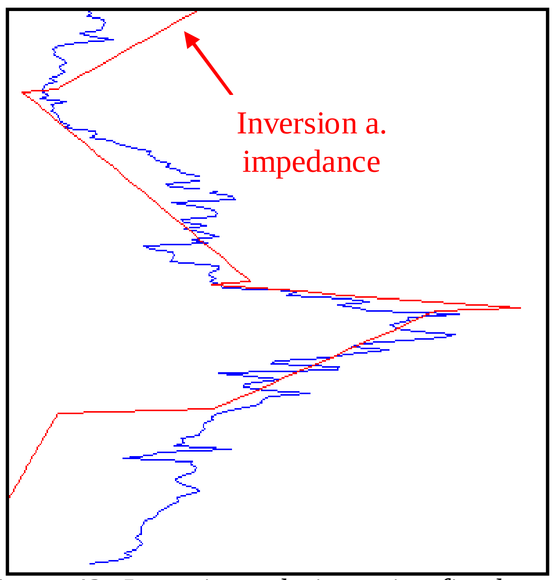


*Figure 43: Inversion solution using five layers.*

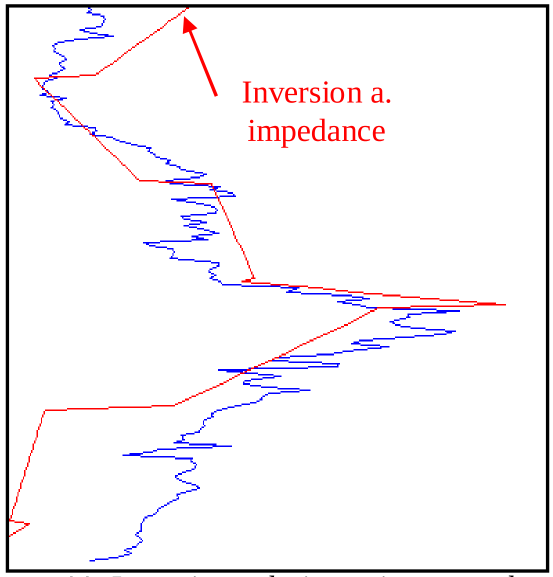



*Figure 44: Inversion solution using seven layers.*

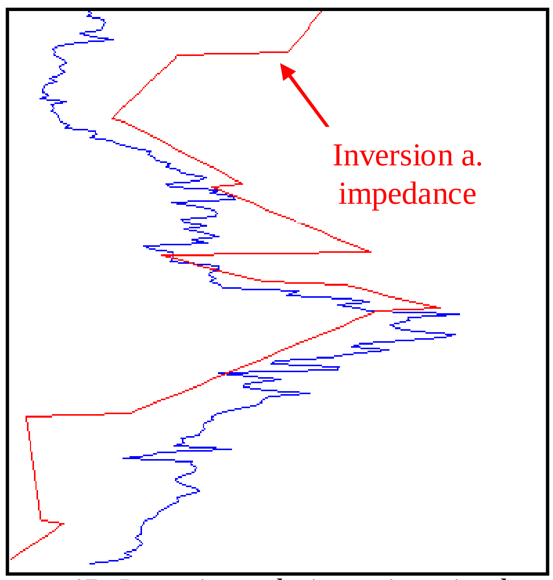



*Figure 45: Inversion solution using nine layers.*

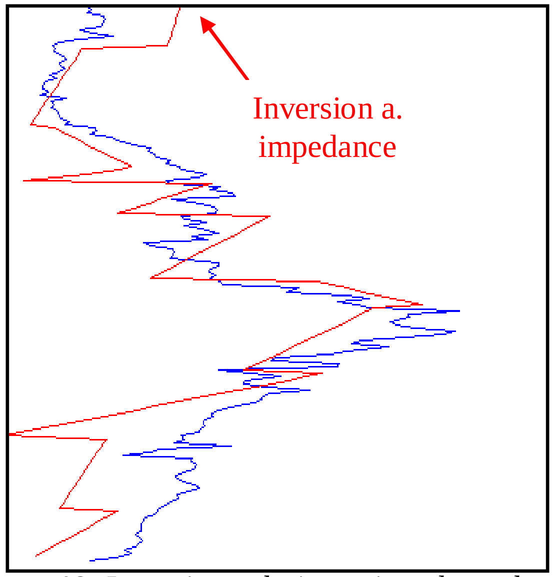



*Figure 46: Inversion solution using eleven layers.*

## 8. Conclusions

Constraining *a priori* information in the inversion algorithm clearly helped in getting the unique solution of the seismic inversion problem, that is, achieving the global minimum of objective function. It was found that one starting acoustic impedance in the model had to be constrained in order to achieve global minimum of the objective function. This conclusion was suggested using the synthetic wedge model (Section 5.4 – Inaccurate constraints), where all constraint values, whether accurate or inaccurate, had the same very weak magnitude of "force", that is, Lagrange multiplier value, $\sim10^{-23}$, making solution non-unique.

However, travel-time thickness of the layer of interest does not have to be constrained to have a non-unique solution, but it is a mean for biasing the inversion in directions found to be desirable for interpretation purposes. It was found in the same synthetic wedge model that when the accurate thickness was constrained, a unique minimum value of Lagrange multiplier was achieved $\sim10^{-23}$ magnitude comparing to the other inaccurate values, making solution unique.

Furthermore, a random sampling technique, often called Monte Carlo sampling, provides the way of scanning complete model space for the sufficiently good initial model.  Each of the solution for each of the initial model has been compared to get one with the minimum error, thus, avoiding the local minimum, to be the solution of the seismic inversion. In the real example at South Timbalier field, it was shown that twenty million trials were sufficient to achieve satisfactory results. Further investigation could be done to computationally determine optimal number of trials that should be used in an inversion process.

The number of layers is crucial in the efficiency of the developed algorithm to get accurate and/or stable result. The more number of layers involved in the inversion, that is, the higher degrees of freedom in the inversion used in the inversion, the closer estimated net-to-gross ratio to the actual net-to-gross ratio. On the other hand, the higher degrees of freedom used in the inversion, the less stable is result, that is, the more difficult is to achieve a non-singular Jacobian in the inversion, and the longer computational time of the inversion. Here, the well-posed solutions, that is, the solutions with the non-singular Jacobian, were achieved with the models with the number of layers between three and twelve. The least-square error with three-layer model global solution was ~2.1, whereas with twelve-layer model global solution was ~0.15, which is improvement in matching actual with predicted data fourteen times.

Travel-time thickness net-to-gross ratio can be good indicator of the quality of reservoir. When the correlation between the acoustic impedance and SP log has been achieved, the estimated net-to-gross ratio gave very good estimation of the layer of interest, even though the layer is below “conventional seismic resolution”, quarter of the dominant wavelength. For the layer of interest in the research, using SP, the net-to-gross ratio was estimated to be 0.85, whereas using the seismic inversion, the “travel-time” net-to-gross was estimated to be 0.87 and 0.89 for ray and effective medium theories, respectively.

However, the estimated gradients of the layers in general should not be used in the interpretation, but on the other hand, the general trend of the estimated gradients has a very good qualitative match with the actual one, that is, the one directly measured in the well.

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