Introduction

In geophysics, it has been common to describe the elastic behavior of rocks in isotropic terms, although most rock masses whose elasticity has been actually measured are anisotropic. A chief reason for this fundamental mismatch between theory and practice was that the equations for elastic anisotropy are much more complex than for isotropy. However, the assumption that the anisotropy is "weak" has proven to be very useful in simplifying the equations for the anisotropic velocities, enabling a more complete description of rock elasticity for seismic analysis. The purpose of the present work is to extend this simplification from the elastic *stiffnesses* (moduli), useful in elastic wave propagation, to the elastic *compliances*, useful in many other contexts.

Elasticity

The linear-elastic constitutive equation for homogeneous anisotropic bodies was first written by Hooke (1678, 1931, 2007); in modern notation it is:

$$\tilde{\tau} = \approx{C}\tilde{\varepsilon} \tag{1}$$

where $\tilde{\tau}$ is the stress (a tensor of 2nd rank), $\tilde{\varepsilon}$ is the strain, and $\overset{\approx}{C}$ is the elastic *stiffness* (modulus) , a tensor of 4th rank. The individual components are given by

$$\tau_{ij} = C_{ijkl}\varepsilon_{kl} \tag{1a}$$

with summation over repeated indices. Equivalently, the same linear relation may be written as

$$\tilde{\varepsilon} = \overset{\approx}{S}\tilde{\tau} \tag{2}$$

where the 4th-rank tensor of coefficients $\overset{\approx}{S}$ is the elastic *compliance* tensor. The linear elastic material properties are entirely expressed by *either* of these tensors (or by certain combinations of their elements).

Because of internal symmetries (Nye, 1985), the 3x3x3x3 tensors may be written as 6x6 matrices, using the so-called Voigt (1928) recipe:

$$\begin{array}{cccccccc} ij & = & 11 & 22 & 33 & 13,31 & 23,32 & 12,21 \\ & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \alpha & = & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \tag{3}$$

With this notation, the elastic stiffness and compliance matrices of *isotropic* bodies may be written as:

$$\overset{\approx}{C} = \begin{vmatrix} c & \lambda & \lambda & & & \\ \lambda & c & \lambda & & & \\ \lambda & \lambda & c & & & \\ & & & \mu & & \\ & & & & \mu & \\ & & & & & \mu \end{vmatrix} \qquad \overset{\approx}{S} = \begin{vmatrix} 1/E & -\nu/E & -\nu/E & & & \\ -\nu/E & 1/E & -\nu/E & & & \\ -\nu/E & -\nu/E & 1/E & & & \\ & & & 1/\mu & & \\ & & & & 1/\mu & \\ & & & & & 1/\mu \end{vmatrix} \tag{4}$$

where μ is the shear modulus (governing shear wave propagation), $c = K + 4\mu/3$ is the longitudinal modulus (governing P-wave propagation), K is the incompressibility, and $\lambda = c - 2\mu$ is the Lame parameter. All elements not shown explicitly are zero. In the compliance tensor, E is Young's modulus, and ν is Poisson's ratio.

Polar anisotropy

The simplest anisotropic case of geophysical interest is that with a single axis of symmetry: polar anisotropy (sometimes called "Transverse Isotropy" [*sic*]). For this case:

$$\tilde{\tilde{C}} = \begin{vmatrix} C_{11} & C_{12} & C_{13} & & & \\ C_{12} & C_{11} & C_{13} & & & \\ C_{13} & C_{13} & C_{33} & & & \\ & & & C_{44} & & \\ & & & & C_{44} & \\ & & & & & C_{66} \end{vmatrix} \qquad \tilde{\tilde{S}} = \begin{vmatrix} S_{11} & S_{12} & S_{13} & & & \\ S_{12} & S_{11} & S_{13} & & & \\ S_{13} & S_{13} & S_{33} & & & \\ & & & S_{44} & & \\ & & & & S_{44} & \\ & & & & & S_{66} \end{vmatrix} \tag{5}$$

Here the evident equivalence of the 1- and 2-axes is a consequence of the assumed rotational symmetry about the polar (3-) axis. There are only 5 independent stiffnesses, since $C_{12} = C_{11} - 2C_{66}$. There are only 5 independent compliances, since $S_{12} = S_{11} - S_{66}/2$. Explicit expressions for the $S_{\alpha\beta}$ in terms of the $C_{\alpha\beta}$, and *vvs*, are given by Nye (1985).

Weak Polar Anisotropy

The equations for velocity at general angles (in terms of elastic stiffnesses), are well-known (cf. e.g., Thomsen , 1986) but are so algebraically complicated that an intuitive understanding of their meaning is difficult. The assumption of weak anisotropy simplifies the equations for elastic wave velocities so markedly that the notation introduced by Thomsen (1986) for polar anisotropy has become conventional (Peltoniemi, 2005). The parameterization in Equation (5) is replaced by

$$V_{P0} \equiv \sqrt{C_{33}/\rho} \qquad V_{S0} \equiv \sqrt{C_{44}/\rho} \tag{6a}$$

$$\varepsilon \equiv \frac{C_{11} - C_{33}}{2C_{33}} \tag{6b}$$

$$\delta \equiv \frac{(C_{13} + C_{44})^2 - (C_{33} - C_{44})^2}{2C_{33}(C_{33} - C_{44})} \tag{6c}$$

$$\gamma \equiv \frac{C_{66} - C_{44}}{2C_{44}} \tag{6d}$$

The non-dimensional parameters (6bcd) all vanish in the limiting case of isotropy, and it is natural to define "weak polar anisotropy" as the case when all are << 1. When the exact equations for the polar-anisotropic velocities are linearized in these small quantities, they simplify substantially (Thomsen, 1986), enabling the wide use of this approximation in analyzing seismic data. However, the definitions themselves (above) do not assume anything about the magnitude of the parameters.

The anisotropy parameters (6bcd) were not found by perturbation theory, but instead were found by inspection from the exact equations; this may be inferred from the quadratic form (6c) for δ. However, if δ is indeed small, then (6c) is equivalent to its linearized (weak-anisotropy) limit:

$$\delta_w \equiv \frac{C_{13} - (C_{33} - 2C_{44})}{C_{33}} \approx \delta \tag{6e}$$

In the case of the elastic wave velocities, no additional simplification results from the use of (6e) in place of (6c). However, in the present case of the compliances, we shall use perturbation theory, and so will require that parameters (6bde) be << 1. This logic, applied to the polar anisotropic compliances, yields the principal result of the present work:

$$\tilde{\tilde{S}} \approx \begin{vmatrix} (1/E_3) + \Delta S_{11} & S_{11} - S_{66}/2 & -\nu_3/E_3 + \Delta S_{13} & & & \\ S_{11} - S_{66}/2 & (1/E_3) + \Delta S_{11} & -\nu_3/E_3 + \Delta S_{13} & & & \\ -\nu_3/E_3 + \Delta S_{13} & -\nu_3/E_3 + \Delta S_{13} & 1/E_3 + \Delta S_{33} & & & \\ & & & 1/C_{44} & & \\ & & & & 1/C_{44} & \\ & & & & & (1-2\gamma)/C_{44} \end{vmatrix} \tag{7}$$

where the following notation is used, as a matter of convenience:

$$K_3 \equiv C_{33} - 4C_{44}/3 \quad ; \quad \lambda_3 \equiv C_{33} - 2C_{44} \quad ; \quad E_3 \equiv \frac{3K_3C_{44}}{C_{33}-C_{44}}; \qquad \nu_3 \equiv \frac{\lambda_3}{2(C_{33}-C_{44})} \tag{8}$$

These are the equivalents of: incompressibility , Lame parameter , Young's modulus, and Poisson's ratio, all calculated from the symmetry-axis stiffnesses. In terms of the familiar anisotropy parameters, the compliance perturbations denoted above are:

$$\Delta S_{11} \equiv \frac{1}{2}\left(\frac{C_{33}}{3C_{44}K_3}\right)^2 \left[-C_{33}\varepsilon + \lambda_3\delta_w + C_{44}\gamma\right] \tag{9a}$$

$$\Delta S_{33} \equiv \frac{2}{E_3}\left(\frac{C_{33}}{3C_{44}K_3}\right)\left[-(C_{33}-E_3)\varepsilon + \lambda_3\delta_w + \frac{C_{44}}{C_{33}}(C_{33}-E_3)\gamma\right] \tag{9b}$$

$$\Delta S_{13} \equiv \frac{2\nu_3}{E_3}\left(\frac{C_{33}}{3C_{44}K_3}\right)\left[C_{33}\varepsilon - \left(\lambda_3 + \frac{3C_{44}K_3}{2\lambda_3}\right)\delta_w - C_{44}\gamma\right] \tag{9c}$$

An example of the effect of these approximations is shown in Figure 1. For each compliance element, the weak anisotropic approximation (7) is *exact* at zero anisotropy, shows the proper *initial slope*, and begins to *deviate* at a value of anisotropy near to 5%.

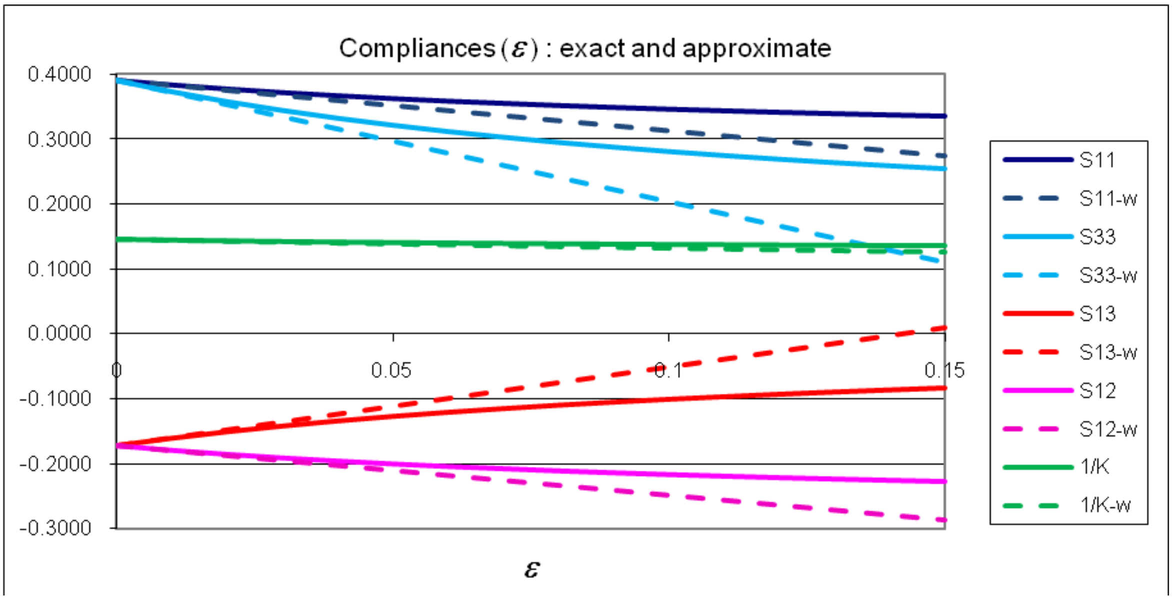

Figure 1 *Four of the compliance elements, and the compressibility, as a function of* $\boldsymbol{\varepsilon}$*, with* $\boldsymbol{\delta_w} = \boldsymbol{\gamma} = 0$*.* C_{33} *and* C_{44} *are fixed at values evident in the graphs. The ratio* $C_{33} / C_{44} = 9$*, corresponding to* $V_{P0} / V_{S0} = 3$*, a typical value for young sedimentary rocks.*

The deviation begins distinctly earlier in a corresponding calculation with δ_w varied, but this is not serious, as δ_w is usually smaller in real rocks than is ε or γ. For older, harder rocks (with smaller values of V_{P0} / V_{S0}), the anisotropic effects are smaller, and the point at which the weak approximation deviates from the exact compliances occurs at larger values of the anisotropies. Also shown in the Figure is the compressibility, discussed next.

Incompressibility

The scalar incompressibility (= bulk modulus = 1/compressibility) is of interest, even in this tensor context. From Equation (2), the dilatation is

$$\frac{\Delta V}{V} = \varepsilon_{ii} = S_{iikl}\tau_{kl} \tag{10}$$

where V is specific volume. If the imposed stress is pressure P, then $\tau_{kl} = -P\delta_{kl}$, and we have

$$\frac{-1}{P}\frac{\Delta V}{V} \equiv \frac{1}{K} = S_{iikk} \tag{11}$$

In terms of the 6x6 matrix elements, this is, for polar anisotropy,

$$K^{-1} = 2S_{11} + 2S_{12} + 4S_{13} + S_{33} \tag{12}$$

To first order in the small anisotropic parameters defined above, this is

$$K \approx K_3 + \frac{8}{9}\left[C_{33}\varepsilon + \frac{C_{33}}{2}\delta_w - C_{44}\gamma \right] \tag{13}$$

In the special case of isotropy, it happens that

$$K_V \equiv \frac{1}{9}C_{iikk} = \frac{1}{9}\sum_{\alpha=1}^{3}\sum_{\beta=1}^{3} C_{\alpha\beta} = K \qquad \text{(isotropy)} \tag{14}$$

where the subscript V indicates that this has the form of a "Voigt-average" (Voigt, 1928) modulus for heterogeneous mixtures.

Of course, for *anisotropic* bodies, $K_V \neq K$, but the question arises: how different are these two quantities? To first order in the small anisotropic parameters defined above, it turns out that

$$K_V \approx K_3 + \frac{8}{9}\left[C_{33}\varepsilon + \frac{C_{33}}{2}\delta_w - C_{44}\gamma \right] \tag{15}$$

which is *identical* (to first order) with equation (13) for K itself. Of course, this particular result applies only to weak *polar* anisotropy, but it would be straightforward to extend the analysis to other symmetry classes. This result helps to explain the experimental result of Berryman and Nakagawa (2009), that for their experiments on stressed sandstones and bead-packs, $K_V \approx K$.

Conclusions

The anisotropy parameters which were defined to simplify the equations for anisotropic elastic velocity are used to analytically describe the (weak) anisotropic variation of the elastic compliances, enabling an understanding of this variation in familiar terms. The scalar incompressibility is identical, to first order in anisotropy, with the average of the elastic stiffnesses given in equation (14).

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References

Berryman, J. G. and S. Nakagawa, 2009. Drained Constants from Undrained Ultrasound Measurements, Inverse Problems in Anisotropic Poroelasticity, submitted to **J. Acoust. Soc. Am**.

Brown, R. J. S. and J. Korringa, 1975. On the dependence of the elastic properties of a porous rock on the compressibility of the pore fluid, **Geophysics**, **40**(4), 608-616.

Hooke, R, 1678. Potentia restitutiva, or Spring, Reprinted in Early science in Oxford (8), 1931. Reprinted in Classics of Elastic Wave Theory, Soc. Expl. Geoph., Tulsa, pp. 55-68, 2007.

Nye, J. F., 1985. Physical Properties of Crystals: Their Representation by Tensors and Matrices, Oxford University Press, Oxford, 352 pp.

Peltoniemi, 2005. Impact factors, citations, and GEOPHYSICS, **Geophysics**, **70**(2), 3MA-17MA.

Thomsen, L., 1986. Weak Elastic Anisotropy, **Geophysics, 51**, 1954-1966.

Voigt, W., 1928. Lehrbuch der Kristallphysik, Teubner, Leipzig, 962 pp.